KiboCUBE Academy

Lecture 16

Introduction to Orbital Mechanics for Microsatellite

Tokyo Metropolitan University

Department of Aeronautics and Astronautics

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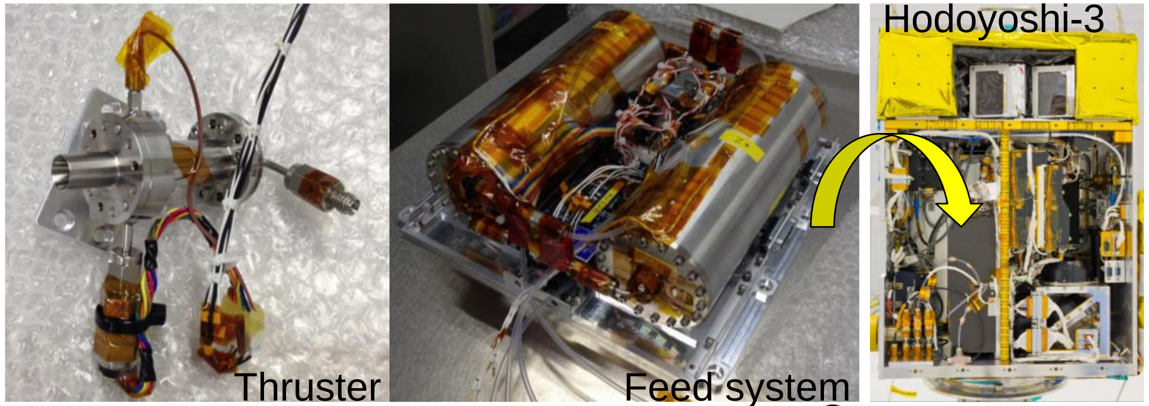
This lecture is NOT specifically about KiboCUBE and covers GENERAL engineering topics of space development and utilization for CubeSats.
The specific information and requirements for applying to KiboCUBE can be found at:
https://www.unoosa.org/oosa/en/ourwork/psa/hsti/kibocube.html

Lecturer Introduction



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SAHARA, Hironori, Ph.D.

Position:

1994	Graduated from Faculty of Engineering, Kyoto University
1996	Master's degree in Engineering from Graduate School of Engineering, Kyoto University
1999	Ph. D from School of Engineering, University of Tokyo
2000 – 2003	Research Fellow, National Aerospace Laboratory of Japan (currently part of JAXA)
2004 – 2007	Research Associate in University of Tokyo
2008 – 2015	Associate Professor in Tokyo Metropolitan University
2016 – present	Professor in Tokyo Metropolitan University

Research Topics:

Development of innovative space systems as propulsion, system architecture, orbit cultivation, and their applications including artificial meteor.

Contents

Introduction to Orbital Mechanics for Microsatellite

1. Basic of Orbital Mechanics

This chapter will take you comprehensively through the basics of orbital mechanics, beginning with the fundamentals.

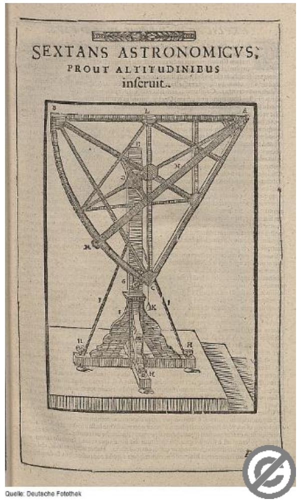
If you understand the contents of this chapter, you have mastered the minimum of orbital mechanics.

1. Basic of Orbital Mechanics

1.1 Kepler's Laws

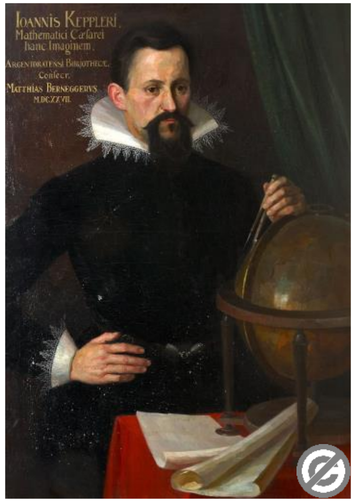
<u>**History of orbital mechanics**</u>

https://en.wikipedia.org/wiki/Tycho_Brahe



- **Tycho Brahe** (1546 – 1601)
  - ✓Made detailed and voluminous observations of planetary movements in an age when telescopes did not yet exist.
- **Johannes Kepler** (1571 – 1603)
  - ✓Attempted to provide theoretical support for the Tycho's observation as an assistant.
  - ✓As a result, a mathematical representation of the observations was successfully obtained:
    - ✓The position and motion of planets could be obtained very accurately by calculation.
    - ✓A force inversely proportional to the square of the distance was shown.
  - ✓Dr. Carl Sagan called him "the first physical astronomer and the last scientific astrologer."

Phenomenon → (induction) → Law



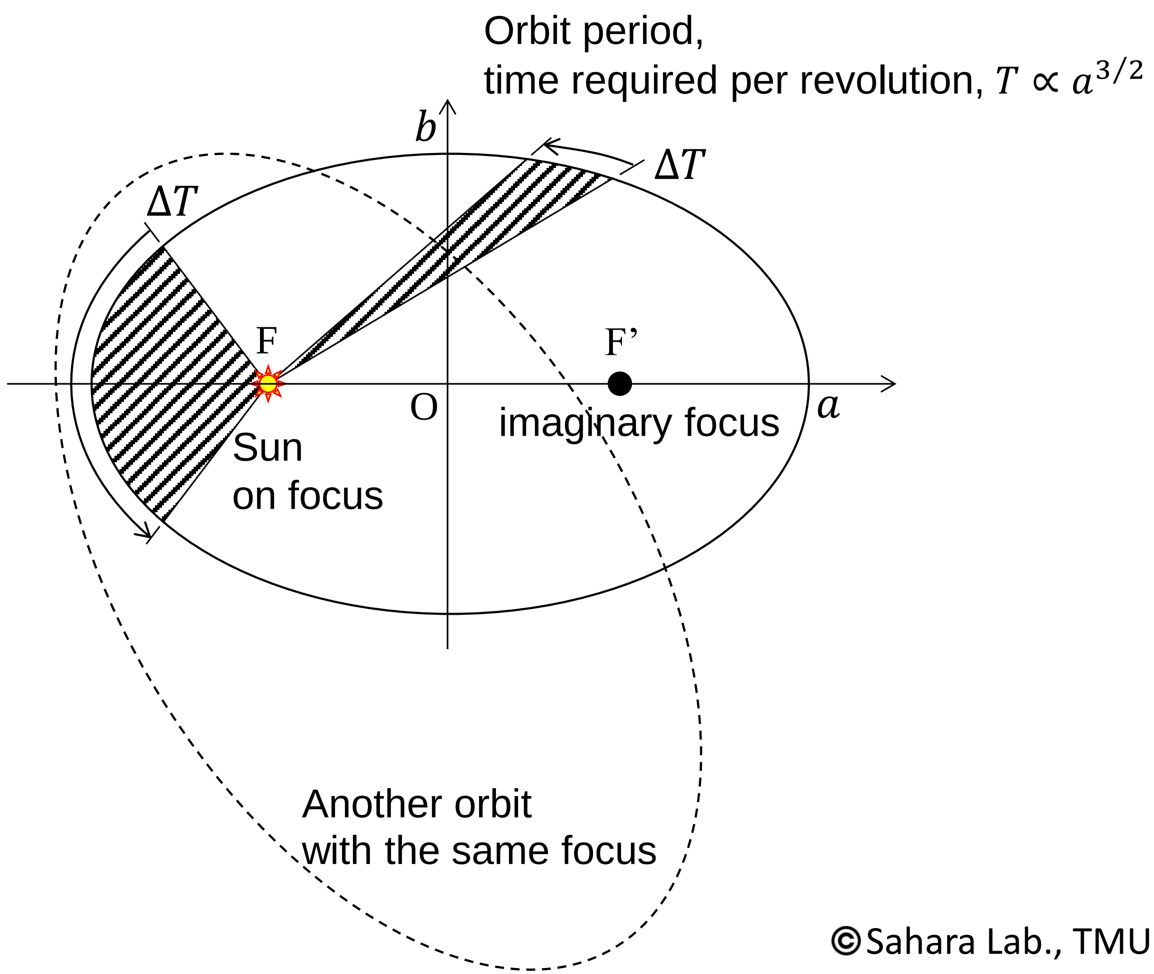
https://en.wikipedia.org/wiki/Johannes_Kepler

Kepler’s Laws

1. **The orbit of a planet is an ellipse with the Sun at one of the two foci.**
   -> This is a description of the "shape" of a planet's orbit.

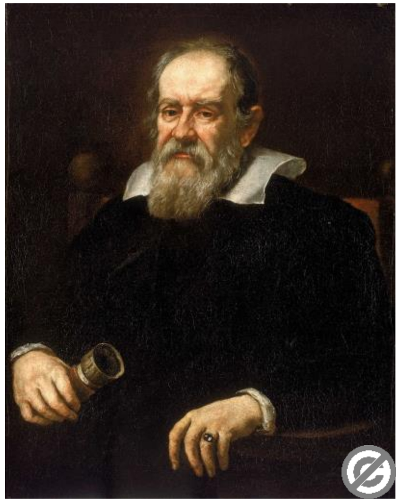
2. **A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.**
   -> This describes the motion "within a single orbit."

3. **The square of a planet's orbital period is proportional to the cube of the length of the semimajor axis of its orbit.**
   -> This clarifies the relationship "between the different orbits."



History of orbital mechanics (cont.)

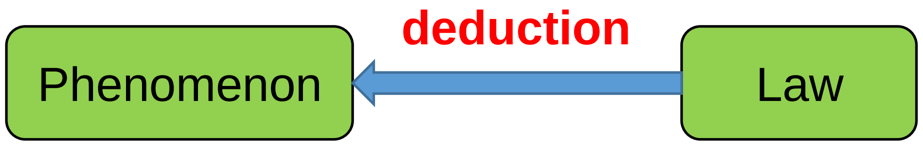
https://en.wikipedia.org/wiki/Galileo_Galilei



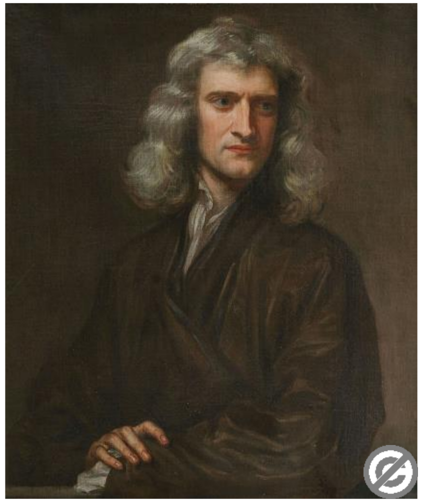
- **Galileo Galilei** (Julian calendar's 1546 – Gregorian calendar's 1601)
  - ✓Is considered the "father of modern science" and "the father of astronomy."
  - ✓Of Kepler's law he said:
    "All celestial bodies move in perfect circles. There is no such thing as elliptical motion."
    It reflects the perceptions and public attitudes of the time that even those who refused to blindly follow the authority of the Church were unable to escape.

- Because of the problem of misalignment with the Julian calendar, the calendar was shifted to the Gregorian calendar with improved leap year rules.
  - Julian calendar : 1 year = 365.25 days
  - Gregorian calendar : 1 year = 365.2425 days
- Promulgated in 1582, the day after October 4 of the same year was designated as October 15.
- NOTE: Julian day (JD) is the number of days from noon (Universal Time) on November 24, 4713 B.C., and is used in astronomy and other fields. In spacecraft operations, J2000.0 (or J2000) is often used.
- In reality, one year = 365.2422 days. If we continue to use the Gregorian calendar, there will be a discrepancy of 0.0003 days per year, or 1 day in 3333 years.

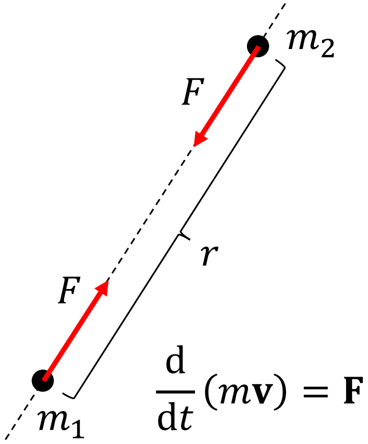
History of orbital mechanics (cont.)



- **Isaac Newton** (1643 – 1727)
  - ✓His achievements, such as the establishment of Newtonian mechanics, the discovery of differential and integral calculus, are nothing short of great.
  - ✓*Hypotheses non fingo* (Latin for "I frame no hypotheses" or "I contrive no hypotheses")
  - ✓He introduced the concept of universal gravitation and applied the laws of motion to successfully describe the motion of the planets.
    It is based on a new style that is not interested in what or why, but accepts it as such.



https://en.wikipedia.org/wiki/Isaac_Newton



$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = \mathbf{F}$$

Newton's laws of motion

1. A body remains at rest, or in motion at a constant speed in a straight line, unless acted upon by a force.
2. When a body is acted upon by a force, the time rate of change of its momentum equals the force.
3. If two bodies exert forces on each other, these forces have the same magnitude but opposite directions.

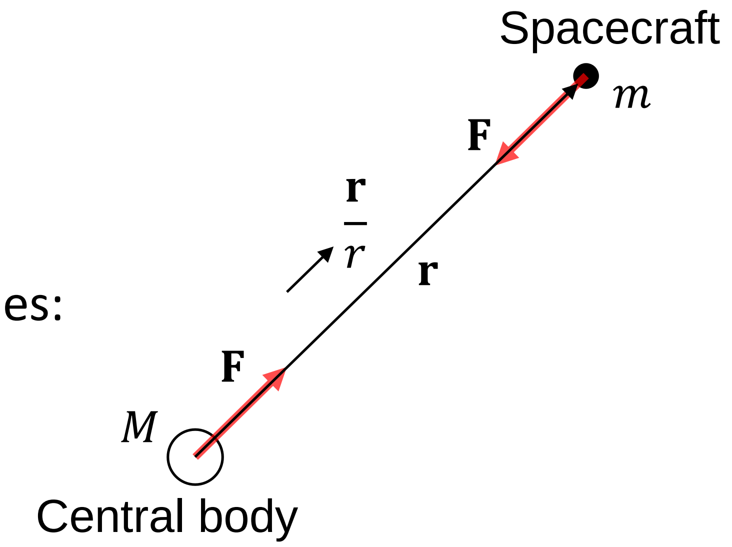
Newton's law of universal gravitation

1. Every point mass attracts every single other point mass by a force acting along the line intersecting both points.
2. The force is inversely proportional to the square of the distance between them.
3. The force is proportional to the product of the two masses.

$$\Longrightarrow \quad F = G\frac{m_1 m_2}{r^2}$$

Integrate the concept of universal gravitation and the laws of motion

- Assume that
  - ✓a two-body problem with a central celestial body and a spacecraft only.
  - ✓the mass of the central body is much larger than the mass of the spacecraft.
- Then, the universal gravitation is a central force field with the following properties:
  - ✓The direction of the force always points toward the central body.
  - ✓The magnitude of the force depends only on the distance to the center of the force.



From the universal gravitation

$$\mathbf{F} = -\frac{GmM}{r^2} \cdot \frac{\mathbf{r}}{r}$$

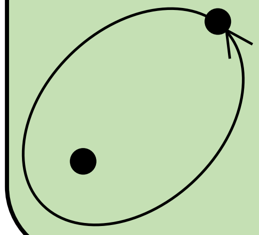
and the equation of motion

$$\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{r}}$$

we obtain the equation of the starting point for orbital mechanics

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0}$$

By the way, if there is no change, a planet (spacecraft) seems to continue on the same stable orbit...



There must be some conserved quantity! Finding conserved quantities is a powerful way to explore the physical phenomena.

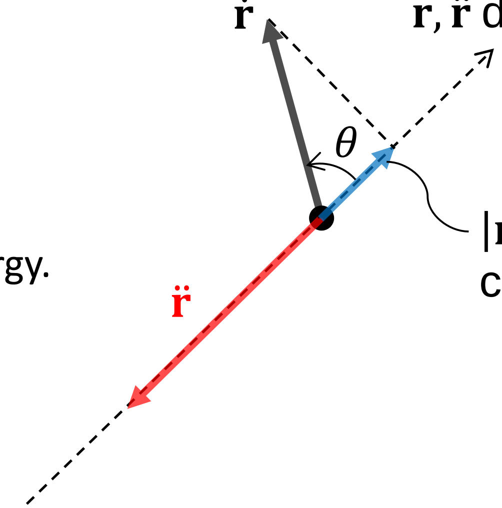
Integrate the concept of universal gravitation and the laws of motion (cont.)

Apply the scalar product of $\dot{\mathbf{r}}$ to both sides of the above equation.

$$\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}}+\dot{\mathbf{r}}\cdot\frac{\mu}{r^3}\mathbf{r}=0$$

[Velocity] x [Force per unit mass in radial direction]v= m/s x N = N-m/s = J/s = W

Scalar



And we found that the above equation expresses the balance with respect to energy.

Applying the following relation

$$\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v^2}{2}\right)\text{ and }\mathbf{r}\cdot\dot{\mathbf{r}}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\cdot\mathbf{r})=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{r^2}{2}\right)$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v^2}{2}\right)+\frac{\mu}{r^3}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{r^2}{2}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v^2}{2}-\frac{\mu}{r}\right)=0$$

When integrated, the result is

$$\frac{v^2}{2}-\frac{\mu}{r}=E$$

E is the integration constant, which is obviously the sum of the kinetic energy and the potential energy per unit mass of the universal gravitation force.

In other words, a specific dynamic energy conservation law was derived.

Dynamic energy is conserved in orbital motion

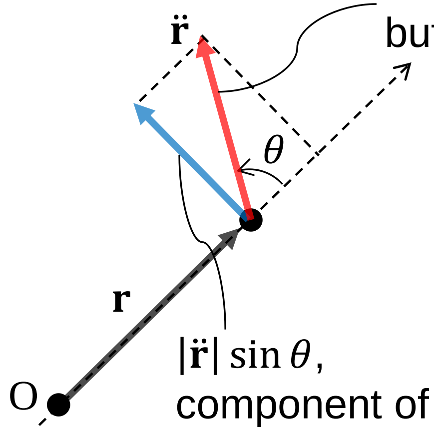
Integrate the concept of universal gravitation and the laws of motion (cont.)

Apply the vector product of $\mathbf{r}$ from the left to both sides of the aforementioned equation.

$$\mathbf{r} \times \ddot{\mathbf{r}} + \mathbf{r} \times \frac{\mu}{r^3}\mathbf{r} = \mathbf{0}$$

[Distance] x [Force per unit mass in circumferential direction] = m x N = N-m

Vector



The above equation expresses the balance with respect to moments.

Applying the following relation

$$\mathbf{r} \times \mathbf{r} = \mathbf{0} \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \dot{\mathbf{r}}) = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \ddot{\mathbf{r}}$$

$= \mathbf{0}$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{0}$$

When integrated, the result is

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{r} \times \mathbf{v} = \mathbf{h}$$

$\mathbf{h}$ is obviously a constant angular momentum vector, and $\mathbf{r}$ and $\mathbf{v}$ always remain in one plane.

That is, the orbit is limited to one plane in space (Kepler's zeroth law).

Angular momentum is conserved in orbital motion.

Example

In the inertial coordinate system, the position and velocity vectors are given as

$$\mathbf{r} = (4.19\mathbf{i} + 6.28\mathbf{j} + 10.46\mathbf{k}) \times 10^6 \text{ [m]}$$

$$\mathbf{v} = (2.59\mathbf{i} + 5.19\mathbf{j}) \times 10^3 \text{ [m/s]}$$

Find the specific dynamic energy E, the specific angular momentum h, and the flight path angle ϕ.
The gravitational constant is $\mu = 3.986 \times 10^5$ [km³/s²].

Answer

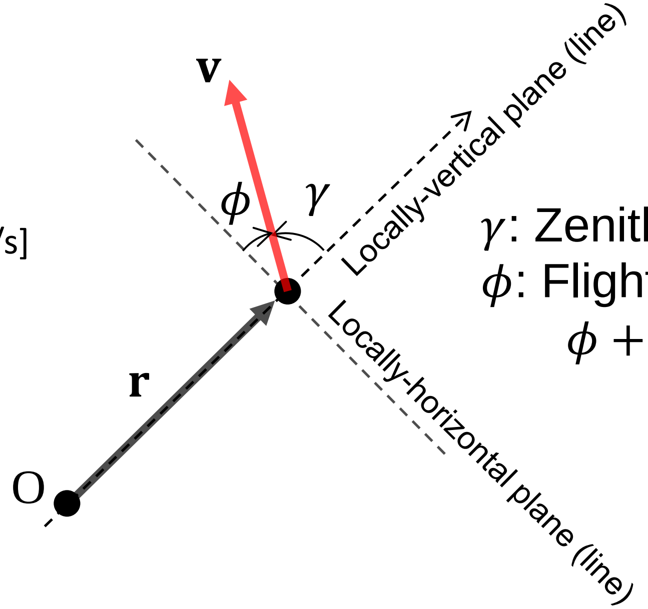
$r = \sqrt{4.19^2 + 6.28^2 + 10.46^2} \times 10^6 = 1.29 \times 10^7$ [m], $v = \sqrt{2.59^2 + 5.19^2} \times 10^3 = 5.80 \times 10^3$ [m/s]

$\therefore E = \frac{v^2}{2} - \frac{\mu}{r} = -1.41 \times 10^7$ [m²/s²]

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4.19 \times 10^6 & 6.28 \times 10^6 & 10.46 \times 10^6 \\ 2.59 \times 10^3 & 5.19 \times 10^3 & 0 \end{vmatrix}$$

$\therefore h = \sqrt{5.43^2 + 2.71^2 + 0.548^2} \times 10^{10} = 6.09 \times 10^{10}$ [m2/s]

$\therefore \phi = \cos^{-1}\frac{h}{rv} = 35.5°$ $\quad \because \mathbf{r} \cdot \mathbf{v} > 0, \quad h > 0 \quad \Longrightarrow \quad 0° \le \phi \le 90°$



γ: Zenith angle
ϕ: Flight-path angle
$\phi + \gamma = 90°$

$$h = |\mathbf{h}| = |\mathbf{r} \times \mathbf{v}| = rv \sin\gamma = rv\cos\phi$$

1.1 Kepler's Laws

Consider what shape the path in the orbit plane will take.

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0}$$

Apply the vector product of the specific angular momentum vector $\mathbf{h}$ to the above equation.

$$\ddot{\mathbf{r}} \times \mathbf{h} + \frac{\mu}{r^3}\mathbf{r} \times \mathbf{h} = \mathbf{0}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{r}} \times \mathbf{h}) = \ddot{\mathbf{r}} \times \mathbf{h} + \dot{\mathbf{r}} \times \frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \ddot{\mathbf{r}} \times \mathbf{h}$$

$= \mathbf{0}$

$$\frac{\mu}{r^3}\mathbf{r} \times \mathbf{h} = \frac{\mu}{r^3}\mathbf{r} \times (\mathbf{r} \times \mathbf{v}) = \frac{\mu}{r^3}\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = \frac{\mu}{r^3}[(\mathbf{r}\cdot\dot{\mathbf{r}})\mathbf{r} - (\mathbf{r}\cdot\mathbf{r})\dot{\mathbf{r}}] = \frac{\mu\dot{r}}{r^2}\mathbf{r} - \frac{\mu}{r}\dot{\mathbf{r}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\mu\frac{\mathbf{r}}{r}\right)$$

$\mathbf{r}\cdot\dot{\mathbf{r}} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\cdot\mathbf{r}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}r^2 = r\cdot\dot{r}$

$\mathbf{r}\cdot\mathbf{r} = r^2$

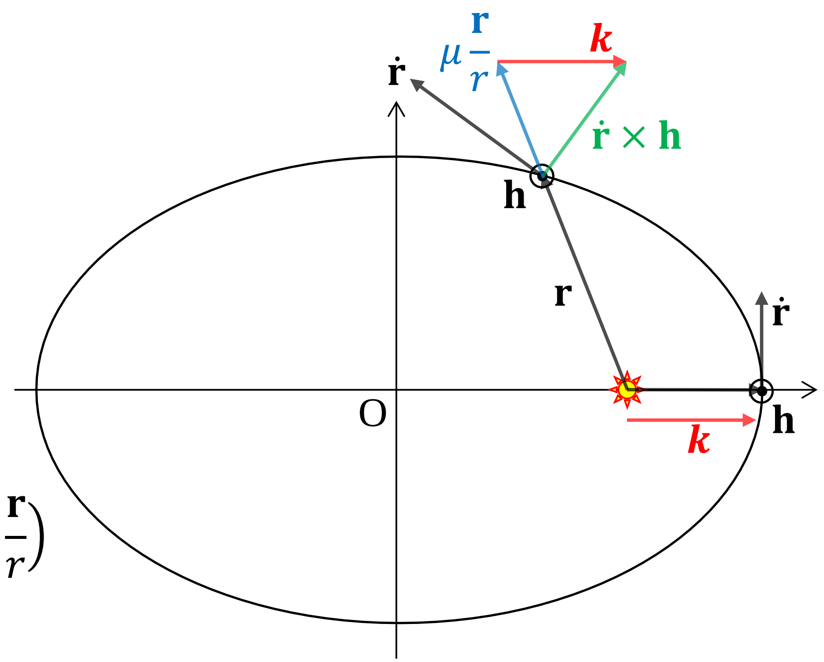
Triple products of vectors
$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

From the above, $\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\mathbf{r}} \times \mathbf{h} - \mu\frac{\mathbf{r}}{r}\right) = \mathbf{0}$

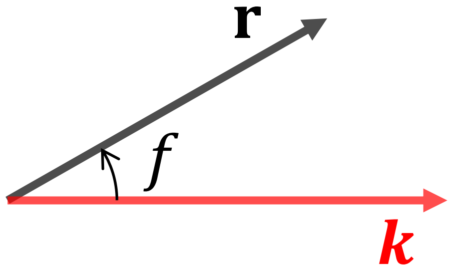
Integrating this, we obtain $\dot{\mathbf{r}} \times \mathbf{h} - \mu\frac{\mathbf{r}}{r} = \mathbf{k}$

The following can be found for $\mathbf{k}$, called the **Laplace vector**.

- $\mathbf{k}$ is in the orbital plane because $\mathbf{h}\cdot\mathbf{k} = 0$.
- $\mathbf{k}$ is a constant value vector because $\mathbf{k}$ is an integral constant.
- $\mathbf{k}$ always indicates the direction of perigee from the central object for any $\mathbf{r}$. Think at periapsis.



The Laplace vector is conserved in orbital motion.



Now that we can make $\mathbf{k}$ a reference direction, let's look at the shape of the orbit using this.

That is, take the scalar product of $\mathbf{k}$ and $\mathbf{r}$.

$$\mathbf{r}\cdot(\dot{\mathbf{r}}\times\mathbf{h})-\mu\frac{\mathbf{r}\cdot\mathbf{r}}{r}=\mathbf{r}\cdot\mathbf{k} \qquad \therefore h^2-\mu r=rk\cos f$$

When rewritten,

$$r=\frac{h^2}{\mu+k\cos f}=\frac{\frac{h^2}{\mu}}{1+\frac{k}{\mu}\cos f}=\frac{p}{1+e\cos f}$$

p : semi-latus rectum
e : eccentricity
f : true anomaly

This represents a quadratic curve (conic curve) expressed in polar coordinates, where f is a parameter.

- When $e=0$, the orbit is **circular** with $E<0$, and $r=p$.
- When $0<e<1$, the orbit is **elliptical** with $E<0$, and $v^2<0$ at $r\to\infty$. In other words, it is trapped in the central body.
- When $e=1$, the orbit is **parabolic** with $E=0$, and $v=0$ at $r\to\infty$. In other words, it just reaches infinity.
- When $1<e$, the orbit is **hyperbolic** with $E>0$, and $v>0$ at $r\to\infty$. It still tries to move away even after reaching infinity.

It's a moment when the mathematical and physical interpretations align so beautifully!!!

This is **Kepler's first law**. However, "elliptical orbit" is extended to "conic curve."

For the implicit function curve

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

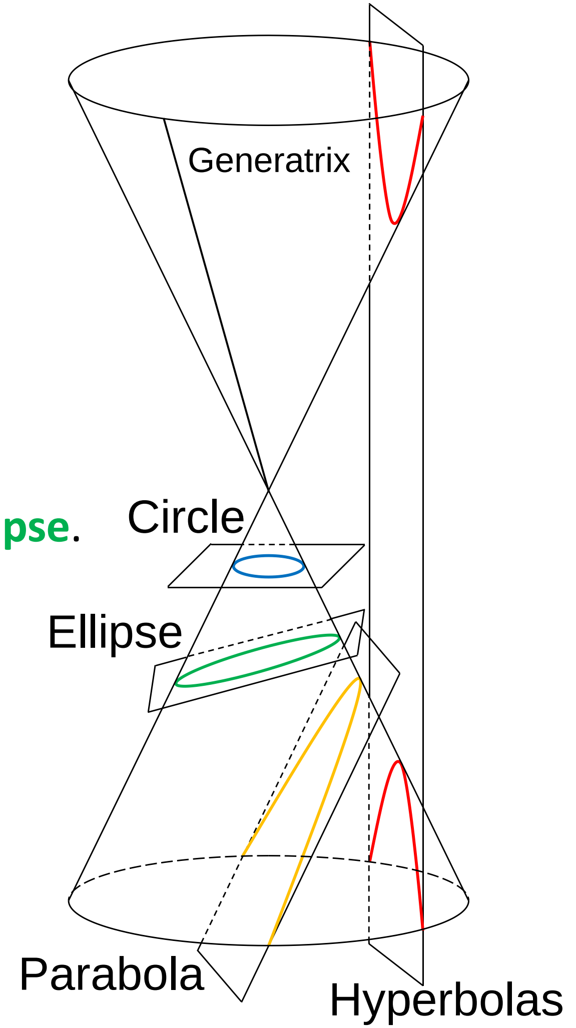
given by the following equation

$$r = \frac{p}{1 + e\cos f}$$

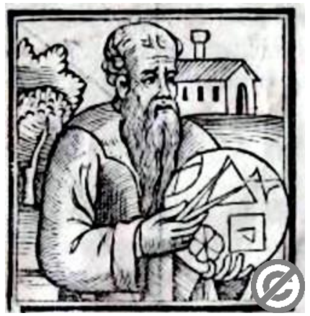
where the radius r is marked in polar coordinates with parameter f.

If a cone is cut in a plane

- that intersects all the generatrix and is parallel to the base, the cross section is a **circle**.
- that intersects all of the basal lines and is not parallel to the base, the cross section will be an **ellipse**.
- that is parallel to a generatrix, the cross section will be a **parabola**.
- that is not parallel to the baseline, the cross section will be a **hyperbola**.



The conic curves were systematized by Apollonius of Perga in BC.



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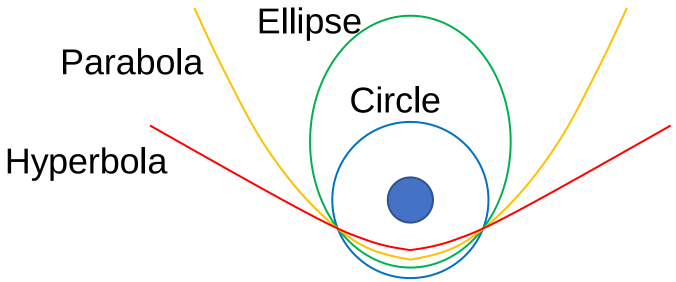


1. Basic of Orbital Mechanics

1.1 Kepler's Laws

The ellipse is one example, but the following holds for all conic curves.
In this lecture, $a < 0$ is used for the hyperbola when $1 < e$.

- From geometric relations,
  - Periapsis $\quad \mathrm{FP} \equiv r_p = a(1-e) = \frac{p}{1+e}$
  - Apoapsis $\quad \mathrm{FA} \equiv r_a = a(1+e) = \frac{p}{1-e}$
  - Semimajor axis $\quad a = \frac{r_a + r_p}{2} = \frac{p}{1-e^2}$
  - Semiminor axis $\quad b = a\sqrt{1-e^2} = \frac{p}{\sqrt{1-e^2}}$

Once any two of semimajor axis, eccentricity, and Semi-latus rectum can be obtained, the remaining one can be obtained, as well as specific dynamic energy, specific angular momentum, etc.

Semi-latus rectum

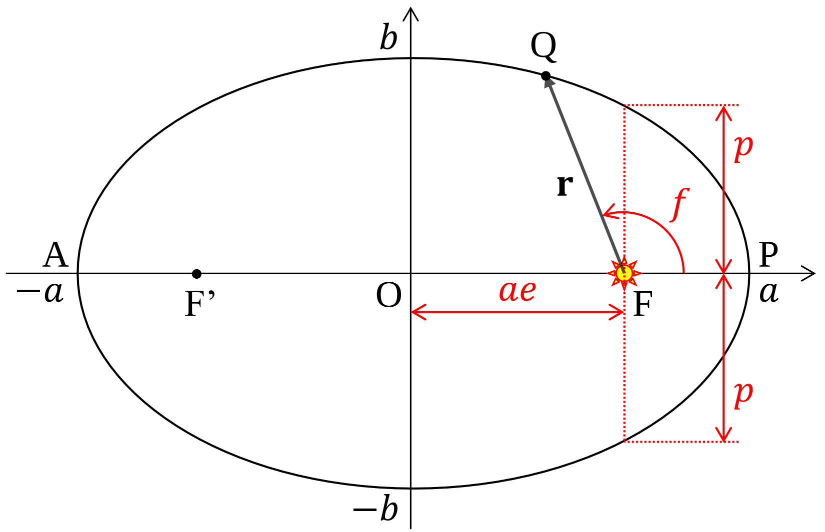
Eccentricity

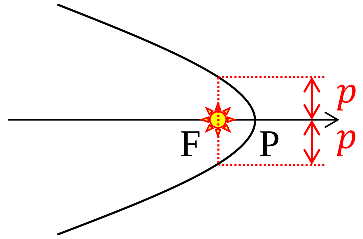
- From the definition in the derivation process, $p = \frac{h^2}{\mu}$, $\quad e = \frac{k}{\mu}$
- Since $h^2 = \mu p = \mu a(1-e^2)$, we can write $h = r_p v_p = r_a v_a$ at Periapsis and Apoapsis, then,

$$E = \frac{v^2}{2} - \frac{\mu}{r} = \frac{h^2}{2r_p^2} - \frac{\mu}{r_p} = \frac{\mu p - 2\mu r_p}{2r_p^2} = \frac{\mu a(1-e^2) - 2\mu r_p}{2r_p^2} = \frac{\mu r_p(1+e) - 2\mu r_p}{2r_p a(1-e)} = \mu\frac{e-1}{2a(1-e)} = -\frac{\mu}{2a}$$

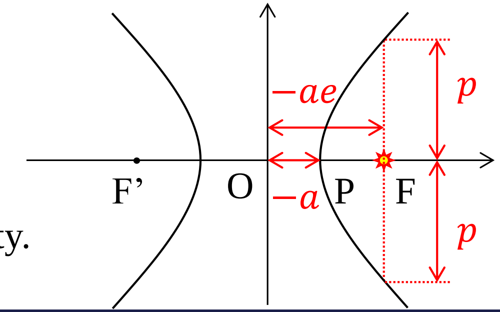
  that is, the specific dynamic energy is uniquely determined once the long radius is determined. And vice versa.

- Eccentricity can be obtained, for example, as follows: $e = \sqrt{1 - \frac{p}{a}} = \sqrt{1 + \frac{2Eh^2}{\mu^2}}$, $\quad e = \frac{r_a - r_p}{r_a + r_p}$





O, A, and F' are at infinity.



A is at infinity.

Velocity is obtained from $E = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$ as $v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$.

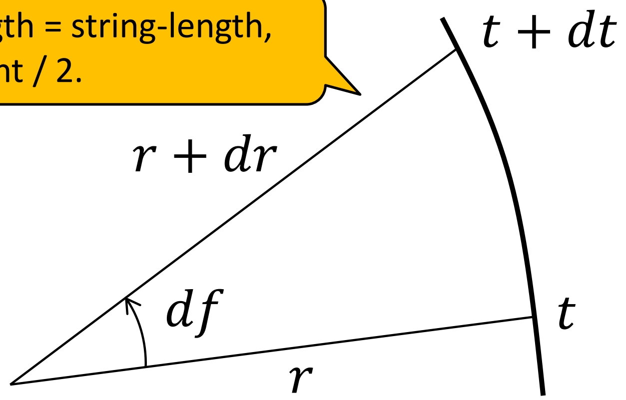
Introducing the concept of flight path angle, we can write as $h = rv\cos\phi = r \cdot r\dot{f} = r^2\frac{\mathrm{d}f}{\mathrm{d}t} \implies dt = \frac{r^2}{h}df$.

By the way, if the angle of motion on an orbit at time dt is df,

the area dA swept by the radius at this time can be written as $dA = \frac{1}{2}r^2 df$ from the concept of a small area.

Eliminating df from the above two equations yields $\frac{dA}{dt} = \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{h}{2}$.

Regarded as a microtriangle (arc-length = string-length, right triangle) and area = base x height / 2.



This is **Kepler's second law**,

and the area velocity is constant at 1/2 of the specific angular momentum.

1. Basic of Orbital Mechanics

1.1 Kepler's Laws

The area of the ellipse is πab.

Since the area velocity of the orbital motion is $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{h}{2}$ from Kepler's second law.

Therefore, the orbital period is the time to fill the entire area of the ellipse, which is $T = \frac{\pi ab}{h/2} = \frac{2\pi ab}{h}$.

Here, since $b = \sqrt{a^2(1 - e^2)} = \sqrt{ap}$ and $h = \sqrt{\mu p}$,

$$T = \frac{2\pi ab}{h} = \frac{2\pi a\sqrt{ap}}{\sqrt{\mu p}} = \frac{2\pi}{\sqrt{\mu}} a^{3/2} \quad \therefore \boxed{T^2 \propto a^3}$$

This is **Kepler's third law**.

First cosmic velocity
is the velocity condition for the spacecraft to be placed into a circular orbit of radius r_c.

- Orbit period $T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$
  For a circular orbit, let $a = r_c$, and $T = \frac{2\pi}{\sqrt{\mu}} r_c^{3/2}$
- Since $v_c = \sqrt{\mu / r_c}$ is obtained from the law of conservation of energy, the first cosmic velocity is $\frac{v_c^2}{2} - \frac{\mu}{r_c} = -\frac{\mu}{2r_c}$.
  Ignoring air drag, the first cosmic velocity at an altitude of 0 km is $v_c = \sqrt{3.986 \times 10^5 / 6378} = 7.905$ [km/s].

Second cosmic velocity
is the escape velocity from the gravitation sphere from a circular orbit of radius r_c.

- Since $E = \frac{v_{esc}^2}{2} - \frac{\mu}{r_c} = 0$ from the law of conservation of energy, $v_{esc} = \sqrt{\frac{2\mu}{r_c}} = \sqrt{2} v_c$.
  Ignoring air drag, the second cosmic velocity at an altitude of 0 km is $v_{esc} = \sqrt{2} v_c = 11.20$ [km/s].

Example

A planetary probe is injected into an Earth escape orbit from a parking orbit at an altitude of 200 km.

Find the minimum escape velocity from the parking orbit and the semi-latus rectum.

Answer

The orbit radius of the parking orbit is $r_c = 200 + 6378 = 6578$ [km].

Therefore,

$$v_{esc} = \sqrt{2\frac{\mu}{r_c}} = \sqrt{\frac{2\times3.986\times10^5}{6578}} = 11.01 \text{ [km/s]}$$

Note that since this is the "minimum" escape velocity, the first of the possible escape trajectories is a parabolic trajectory.
Also, the injection point is the perigee.

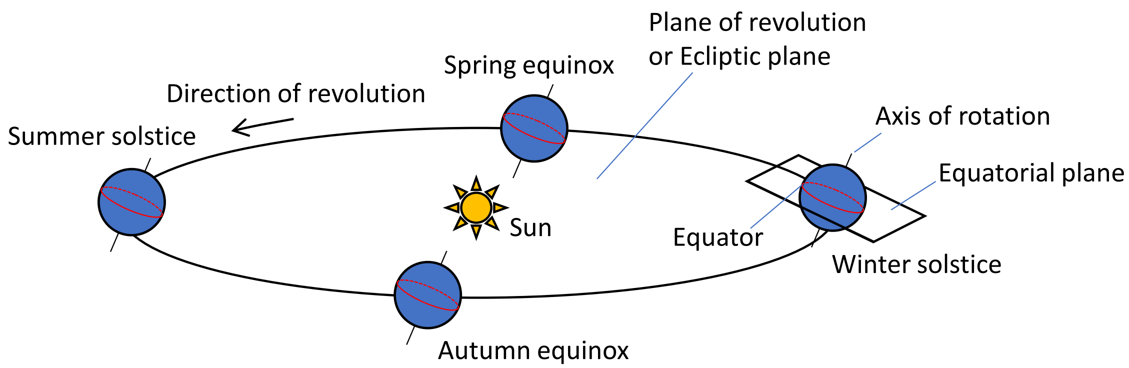
and

$$p = r_p(1+e) = 2r_p = 2\times6578 = 13156 \text{ [km]}$$

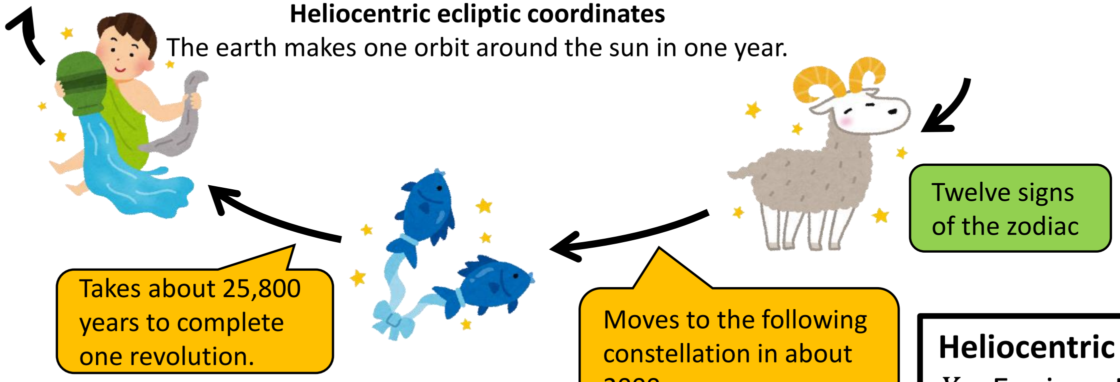
1. Basic of Orbital Mechanics

1.2 Orbit Elements and Orbit Determination

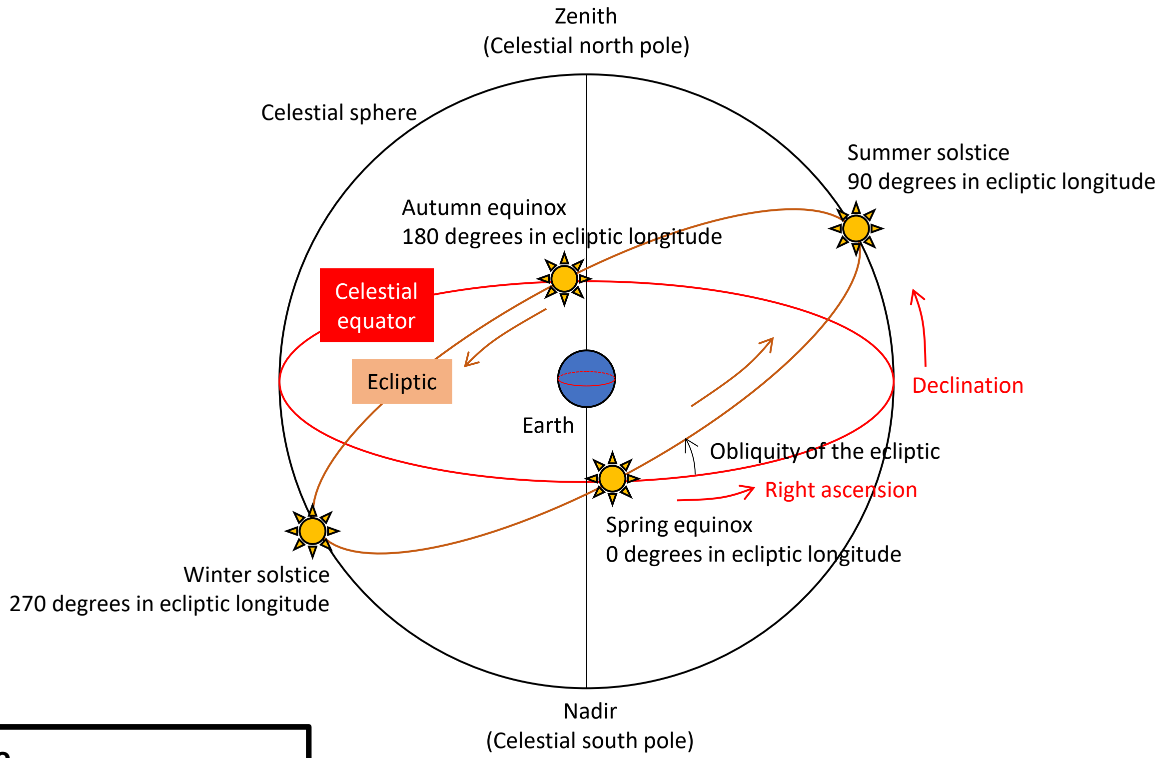
Define an appropriate inertial coordinate system, the perspective of an observer in inertial motion (stationary or constant velocity linear motion), to represent the trajectory in 3-dimensional space.



Heliocentric ecliptic coordinates
The earth makes one orbit around the sun in one year.



Heliocentric ecliptic coordinate
X_{ϵ}: Equinox Point Direction
Y_{ϵ}: Right-handed system for X_{ϵ} in the ecliptic plane
Z_{ϵ}: Right-hand system for X_{ϵ}-Y_{ϵ}



Display on the celestial sphere
The sun makes one revolution of the ecliptic per year.

ECI: Earth−Centered Inertial

X: Equinox Point Direction

As is customary, it is taken in the direction of Aries, ♈*.*

Y: Right-handed system for X in the equatorial plane

Z: Right-hand system for X-Y

Note that the unit vectors in the X, Y, and Z directions are written as $\boldsymbol{I}$, $\boldsymbol{J}$, and $\boldsymbol{K}$, respectively, in this lecture.

In reality, it is not an inertial system due to its orbit and precession, but it can be regarded as an inertial system as an approximation due to its proximity to the Earth and short time period.

Ground surface center-horizontal plane inertial coordinate system

Origin: Observation point (topos)

X_h: Facing south from the origin

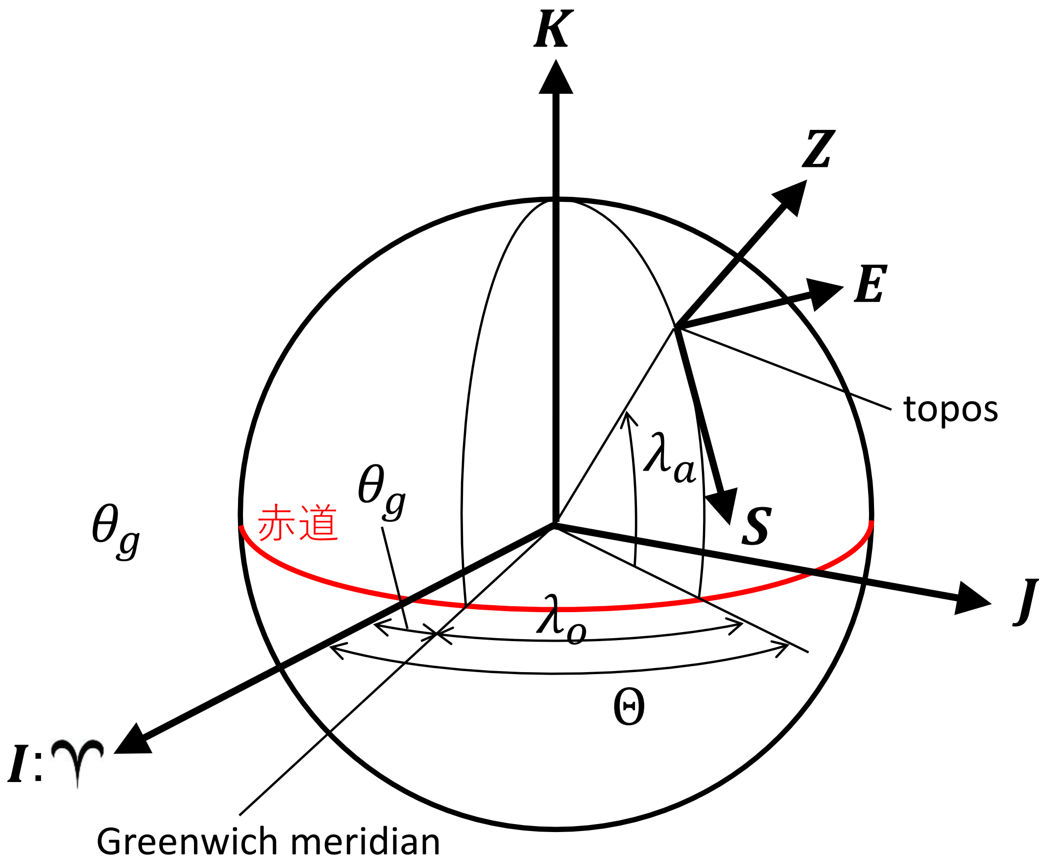
Y_h: Facing east from the origin

Z_h: Right-hand system for X_h-Y_h (local horizontal plane)

Note that the unit vectors in the X_h, Y_h, and Z_h directions are written as $\boldsymbol{S}$, $\boldsymbol{E}$, and $\boldsymbol{Z}$, recpectively, in this lecture.

Observations will be made in the $\boldsymbol{SEZ}$ system.

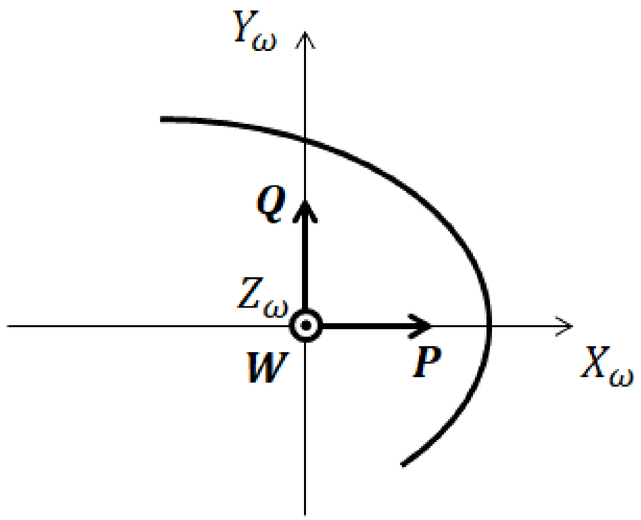
Convert to the $\boldsymbol{IJK}$ system considering topos λ_a, λ_o, and the time of the Earth ($\Theta = \theta_g + \lambda_o$).

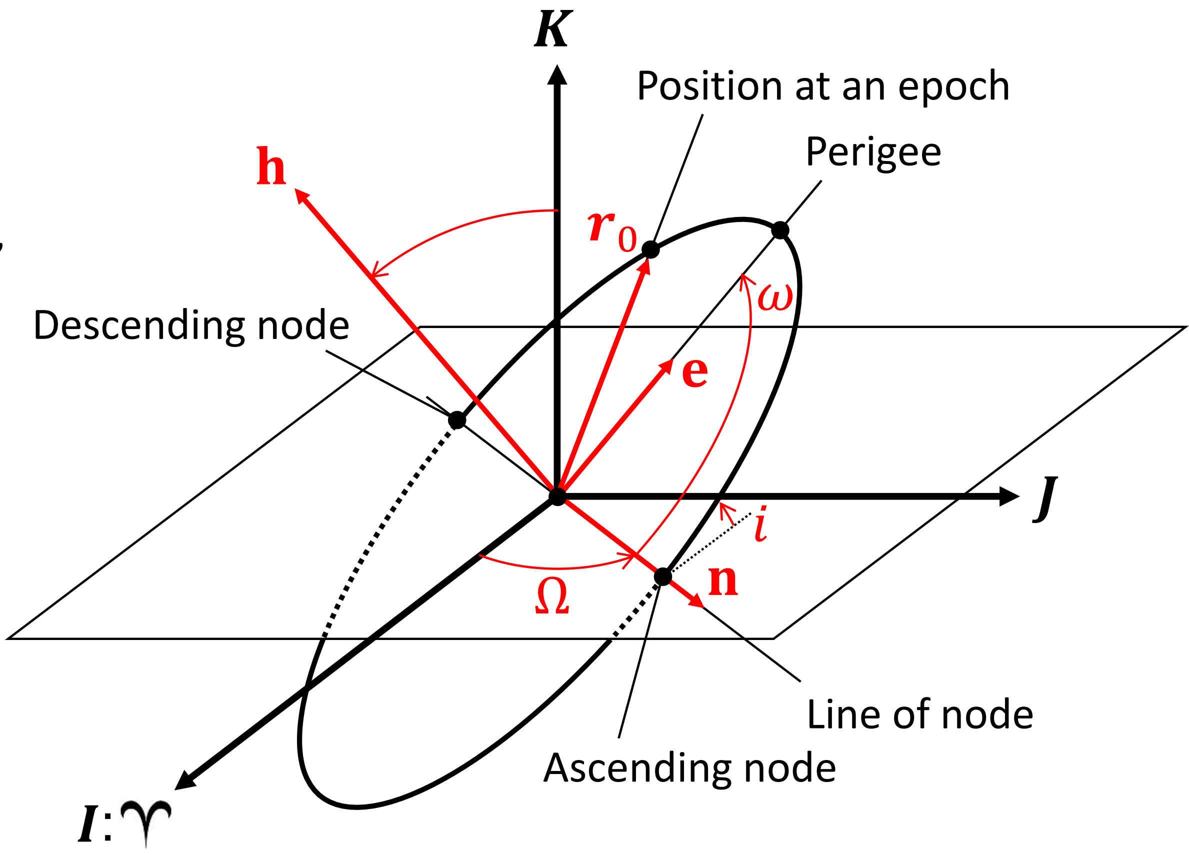


1.2 Orbit Elements and Orbit Determination

Orbital plane coordinate system
X_ω: Periapsis
Y_ω: Right-handed system for X_ω in the orbital plane
Z_ω: Right-hand system for X_ω-Y_ω
Note that the unit vectors in the X_ω, Y_ω, and Z_ω directions are written as $\boldsymbol{P}$, $\boldsymbol{Q}$, and $\boldsymbol{W}$, respectively, in this lecture.

Since it is based on the orbital plane of satellite, it is unaware of the epoch (in the direction of Spring equinox, for example) or the rotation of the Earth.



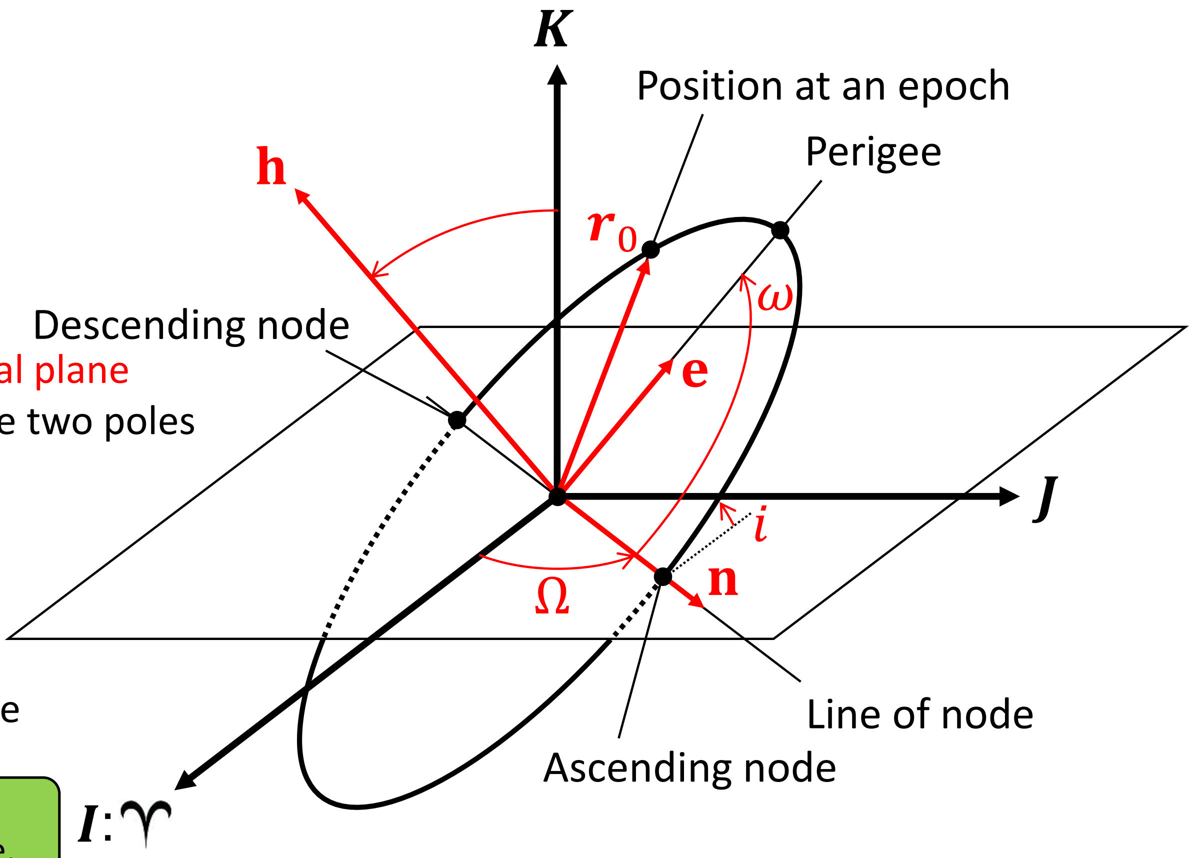


Orbit Elements

- To uniquely determine an orbit in 3-dimensional space, five independent parameters are needed that describe the size, shape, and 3-dimensional orientations of the orbit.
- One more parameter is needed to indicate its position in the orbit.

The above are collectively referred to as the **six orbital elements**.

1. **Semimajor axis**, a : semimajor axis of ellipse
2. **Eccentricity**, e : eccentricity of ellipse
3. **Inclination**, i : angle between orbital plane and equatorial plane
   $i = 0°$: over the equator, $i = 90°$: over the two poles
4. **Right ascension of the ascending node (RAAN)**, Ω
   : longitudinal angle of the ascending node in the equatorial plane
   from the direction of the spring equinox
5. **Argument of perigee**, ω
   : Angle of geocentricity from ascending node to perigee in the orbit plane



Semimajor axis and eccentricity determine the size and shape of the orbit,
and Inclination, RAAN, and argument of perigee are the Eulerian angles of the orbit plane.

6. Specified by **the time, t_0, when the satellite passed its perigee**, or **the true anomaly, f_0, in an epoch**, and so on.

Example

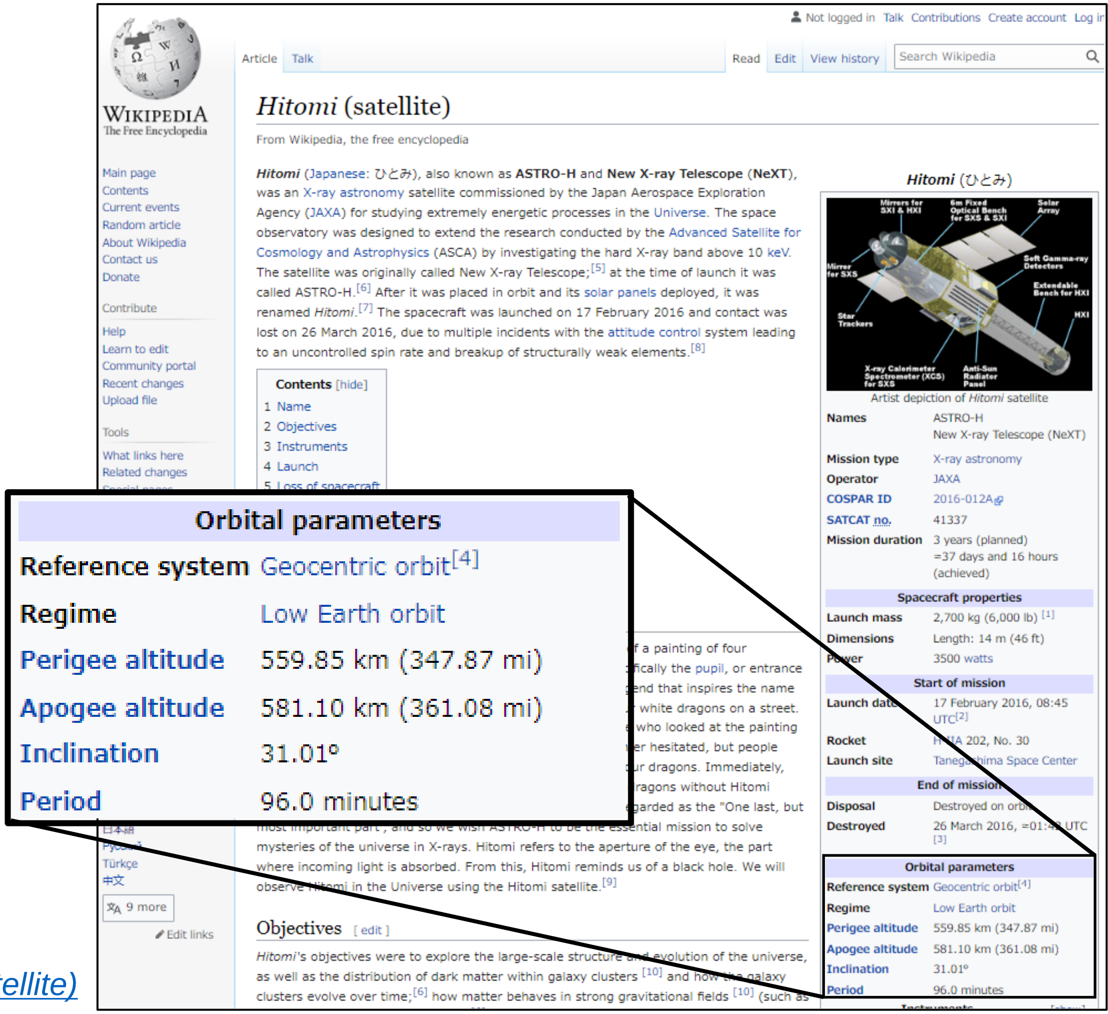
Find the orbit elements of Hitomi, a Japanese X-ray astronomy satellite.
Here, assume that radius of the Earth is 6,378 km in radius.

Answer

The following are obtained from the data on the right.

Perigee radius, $r_p = 559.85 + 6378 = 6937.85$ km,
and apogee radius, $r_a = 581.10 + 6378 = 6959.1$ km, then,

- Semimajor axis: $a = \frac{6959.1+6937.85}{2} \simeq 6948.5$ km
- Eccentricity: $e = \frac{6959.1-6937.85}{6959.1+6937.85} \simeq 0.0015$
- Inclination: $i = 31.01$ deg.



https://en.wikipedia.org/wiki/Hitomi_(satellite)

Two Line Elements, TLE

In actual satellite operations, it is more common to treat the orbital six elements as TLEs than to look at them directly.
TLE is available from **Space-Track.Org** (or **CELESTRACK**), etc.

AAAAAAAAAAAAAAAAAAAAAAAA
1 BBBBBC DDEEEFFF GGHHH.HHHHHHHH +.IIIIIIII +JJJJJ-J +KKKKK-K L MMMMN
2 BBBBB PPP.PPPP QQQ.QQQQ RRRRRRR SSS.SSSS TTT.TTTT UU.UUUUUUUUVVVVVW

Line 1

GG : last two digits of the year of the latest epoch
HHH.HHHHHHHH : the latest epoch (cont.), time in days elapsed since 00:00 UTC on January 1 of the year indicated by GG
L : orbit model used (0: no information, 1: SGP, 2: SGP4, 3: SDP4, 4: SGP8, 5: SDP8)
MMMM : serial number of orbit element (+1 per renewal)

Line 2

PPP.PPPP : inclination (deg.), i
QQQ.QQQQ : RAAN (deg.), Ω
RRRRRRR : eccentricity (decimal point), e
SSS.SSSS : argument of perigee (deg.), ω
TTT.TTTT : mean anomaly (deg.)
UU.UUUUUUUU : mean motion (revolution per day), $n = \frac{2\pi[\text{rad}]}{T[\text{day}]} = \sqrt{\frac{\mu}{a^3}}[\text{rad/day}] \longrightarrow a$

The time elapsed since the perigee passage is expressed as a percentage of the orbital period, $M - M_0 = n(t - t_0)$.
The current position is obtained by solving the Kepler equation, $M = E - e\sin E$.

Example: XI-IV, one of the world's first CubeSats launched by the University of Tokyo

CUBESAT XI-IV (CO-57) 13.37:47.5..., Feb. 7, 2021
1 27848U 03031J 21038.56791106 .00000056 00000-0 45308-4 0 9990
2 27848 98.6882 49.3064 0010811 106.4206 253.8161 14.21866761913357 $n = \sqrt{\frac{\mu}{a^3}} = 14.21866761913357[\text{rad/day}] \longrightarrow a = 7{,}197[\text{km}]$

(98.6882: i; 49.3064: Ω; 0010811: e; 106.4206: ω)

HHH.HHHHHHHH has the value of 001.00000000 on Jan. 1 (UTC).
Note that 000.00000000 means UTC00:00 on Dec. 31 in the previous year.

Orbit Determination, #1

Determine the orbital elements from the position and velocity.

Suppose that at a certain time t, the position $\mathbf{r}$ and velocity $\mathbf{v}$ of a spacecraft in the ECI system are obtained by radar observation, etc. The following three are immediately obtained.

- Specific angular momentum vector

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ r_i & r_j & r_k \\ v_i & v_j & v_k \end{vmatrix} = h_i\mathbf{I} + h_j\mathbf{J} + h_k\mathbf{K}$$

$h_i = r_j v_k - r_k v_j$, etc.

$\mathbf{h}$ is orthogonal to the orbital plane

- Line-of-node vector

$$\mathbf{n} = \mathbf{K} \times \mathbf{h} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 0 & 0 & 1 \\ h_i & h_j & h_k \end{vmatrix} = -h_j\mathbf{I} + h_i\mathbf{J}$$

$\mathbf{n}$ is orthogonal to $\mathbf{K}$ and $\mathbf{h}$, because $\mathbf{n}$ is contained in both the equatorial and orbital planes.

- Eccentricity vector

$$\mathbf{e} = \frac{\mathbf{k}}{\mu}$$

This has a perigee direction from the center of the Earth, and vector whose magnitude is equal to the eccentricity.

The following parameters are obtained one after another.

$$p = \frac{|\mathbf{h}|^2}{\mu}, \quad e = |\mathbf{e}|, \quad a = \frac{p}{1 - e^2}$$

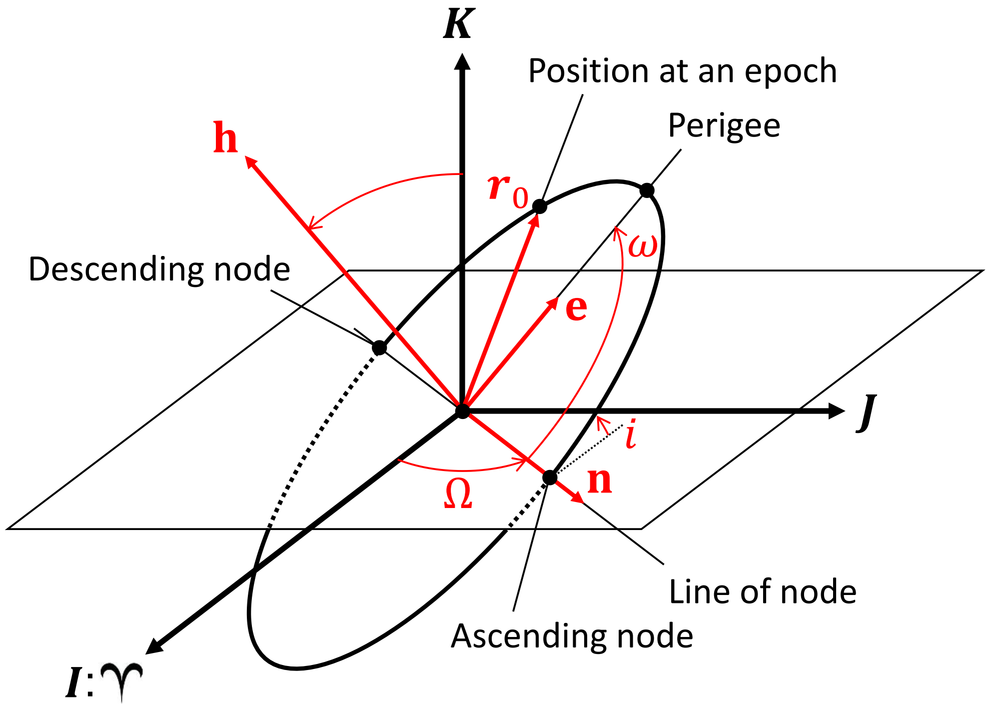
i is the corner between $\mathbf{K}$ and $\mathbf{h}$: $\cos i = \frac{\mathbf{K} \cdot \mathbf{h}}{h} = \frac{h_k}{h}$

Ω is the corner between $\mathbf{I}$ and $\mathbf{n}$: $\cos \Omega = \frac{\mathbf{I} \cdot \mathbf{n}}{n} = \frac{n_i}{n}$

ω is the corner between $\mathbf{n}$ and $\mathbf{e}$: $\cos \omega = \frac{\mathbf{n} \cdot \mathbf{e}}{ne}, \quad \sin \omega = \frac{|\mathbf{n} \times \mathbf{e}|}{ne}$

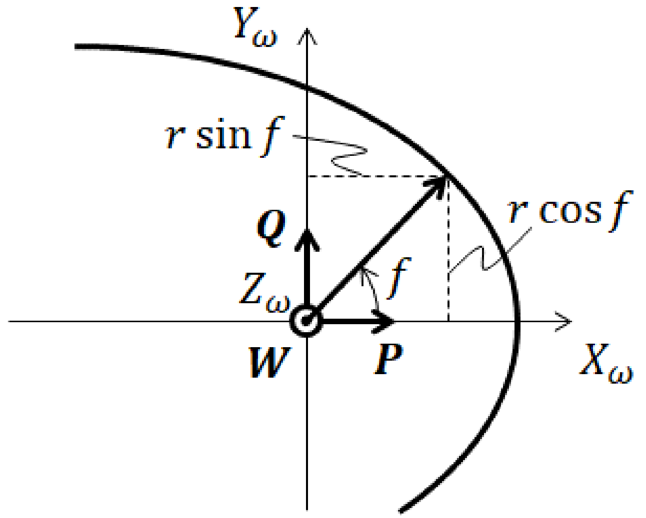
f_0 is the corner between $\mathbf{e}$ and $\mathbf{r}_0$: $\cos f_0 = \frac{\mathbf{e} \cdot \mathbf{r}_0}{e r_0}$

From $\mathbf{n} \times \mathbf{e} = ne \sin \omega \cdot$**P**,
P is the unit vector perpendicular to the orbital plane.
(see the orbital plane coordinate system (**PQW** system))



<u>Orbit Determination, #2</u>

Determine the position and velocity from the orbital elements..



Suppose that orbit six elements, p, e, i, Ω,ω, f, is obtained.

In the orbital plane coordinate system, $\mathbf{r} = r\cos f\,\mathbf{P} + r\sin f\,\mathbf{Q}$

Differentiating this yields $\dot{\mathbf{r}} = \mathbf{v} = \left(\dot{r}\cos f - r\dot{f}\sin f\right)\mathbf{P} + \left(\dot{r}\sin f - r\dot{f}\cos f\right)\mathbf{Q}$

From $r = p(1 + e\cos f)^{-1}$ and $r^2\dot{f} = h = \sqrt{\mu p}$,

$$\dot{r} = p\cdot(-1)\cdot(1 + e\cos f)^{-2}\cdot e\dot{f}(-\sin f) = \frac{p}{1+e\cos f}\cdot\frac{p}{1+e\cos f}\cdot\frac{e\dot{f}\sin f}{p} = \frac{r^2 e\dot{f}\sin f}{p} = \frac{he\sin f}{p} = \frac{\sqrt{\mu p}e\sin f}{p} = \sqrt{\frac{\mu}{p}}e\sin f$$

$$r\dot{f} = \frac{h}{r} = \frac{\sqrt{\mu p}}{r} = \sqrt{\mu p}\cdot\frac{1+e\cos f}{p} = \sqrt{\frac{\mu}{p}}(1 + e\cos f)$$

Then,

$$\mathbf{v} = \left[\sqrt{\frac{\mu}{p}}e\sin f\cos f - \sqrt{\frac{\mu}{p}}(1 + e\cos f)\sin f\right]\mathbf{P} + \left[\sqrt{\frac{\mu}{p}}e\sin f\cos f + \sqrt{\frac{\mu}{p}}(1 + e\cos f)\cos f\right]\mathbf{Q} = \sqrt{\frac{\mu}{p}}[-\sin f\,\mathbf{P} + (e + \cos f)\mathbf{Q}]$$

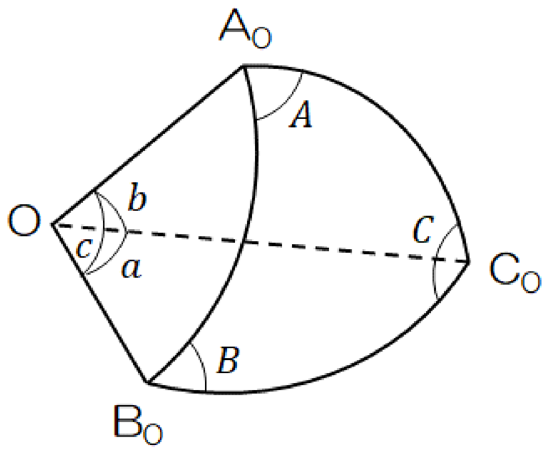
Characteristic Orbit, #1

$\triangle\, \mathrm{A_0B_0C_0}$ is a spherical triangle on the sphere centered at O with $\mathrm{OA_0} = \mathrm{OB_0} = \mathrm{OC_0}$.
Now, if $\mathrm{C} = 90°$, it is a right spherical triangle and the following holds:

$$\sin A = \frac{\sin a}{\sin c} = \frac{\cos B}{\cos b}, \quad \cos A = \frac{\tan b}{\tan c}, \quad \tan A = \frac{\tan a}{\sin b}$$

$$\sin B = \frac{\sin b}{\sin c} = \frac{\cos A}{\cos a}, \quad \cos B = \frac{\tan a}{\tan c}, \quad \tan B = \frac{\tan b}{\sin a}$$

$$\cos c = \cos a \cos b = \cot A \cot B$$

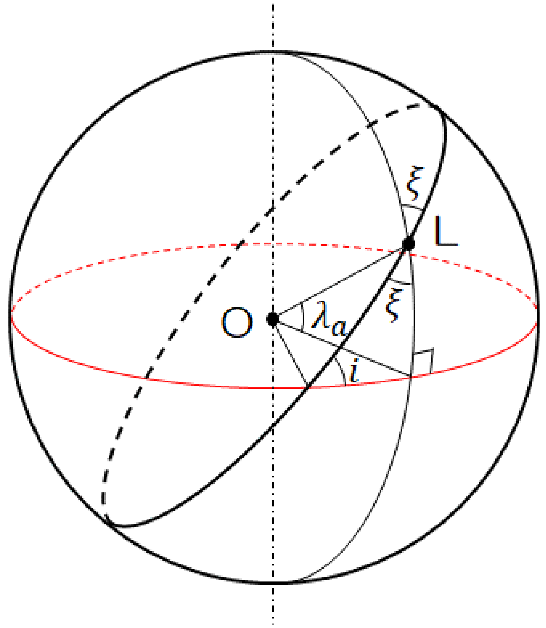


If we launch with an azimuth angle of ξ with respect to north from a launch point (L) at latitude of λ_a, we get

$$A = \xi, \quad B = i, \quad C = \frac{\pi}{2}, \quad b = \lambda_a$$

$$0 \le \lambda_a \le \frac{\pi}{2} \quad \Longrightarrow \quad 1 \ge \cos\lambda_a \ge 0, \quad 0 \le \xi \le \pi \quad \Longrightarrow \quad 0 \le \sin\xi \le 1$$

then, $\sin\xi = \dfrac{\cos i}{\cos\lambda_a} \quad \Longrightarrow \quad \cos i = \cos\lambda_a \sin\xi \le \cos\lambda_a.$



Therefore, $i \ge \lambda_a$ meaning that **inclination cannot be smaller than launch point latitude**.

1. Basic of Orbital Mechanics

1.2 Orbit Elements and Orbit Determination

Characteristic Orbit, #2

- **Launch due east**
  When selected $\xi = \frac{\pi}{2}$, it becomes launch due east and $i = \lambda_a$.
  This is the direction in which Earth's rotation speed ($v_{30°} = 0.403$ [km/s], $v_{45°} = 0.329$ [km/s]) can be used most efficiently. The launch vehicle accelerates a smaller amount of fuel, which reduces the amount of fuel carried and increases weight of payload. This is often the case with astronomical observation satellites, which need to carry many observation instruments.

  Latitude of the world's launch sites

  - Guiana Space Centre (ESA) at 5 degrees and 3 minutes north latitude
  - Christmas Island (NASDA, former a part of JAXA) at 1 degrees and 53 minutes north latitude
  - John F. Kennedy Space Center (NASA) at 28 degrees and 31 minutes north latitude

The reasons for having a launch site in a lower-latitude region are as follows:
1. To secure the degree of freedom of inclination of an orbit by launch azimuth angle.
2. To use the Earth's rotation speed.

- **J_2 term**

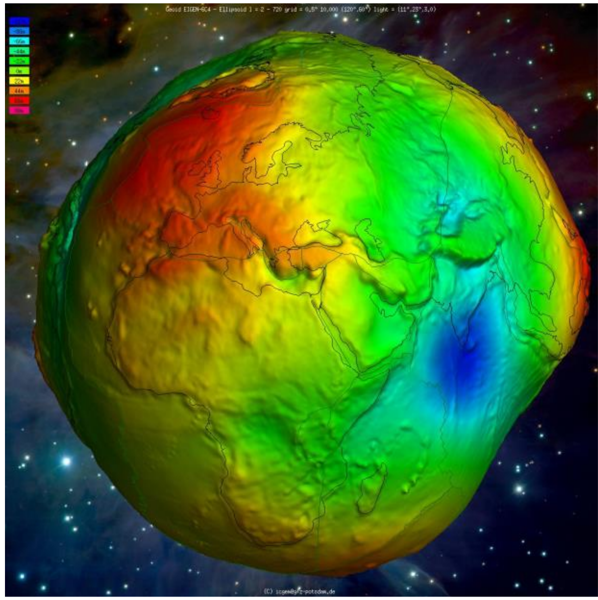
  Since the Earth is not truly spherical and its density distribution is not spherically symmetric, the Earth's gravity field is not spherically symmetric.

$J_l = -C_{l0}$ | Legendre polynomial | Associated Legendre function of the first kind

$$U(r) = -\frac{\mu}{r}\left[1 - \sum_{l=2}^{\infty} J_l \left(\frac{R_\oplus}{r}\right)^l P_l(\sin\varphi) + \sum_{l=2}^{\infty}\sum_{m=0}^{l} J_{lm}\left(\frac{R_\oplus}{r}\right)^l P_{lm}(\sin\varphi)\cos m(\lambda - \lambda_{lm})\right]$$

$$J_{lm}^2 = C_{lm}^2 + S_{lm}^2, \quad \lambda_{lm} = \left[\tan^{-1}\frac{S_{lm}}{C_{lm}}\right]/m$$

C_{lm}, S_{lm} : Coefficients of the nth-degree mth-order spherical harmonics



Displacement from the geoid
http://icgem.gfz-potsdam.de/

Assuming axisymmetry here, the longitudinal distribution can be neglected.

$$U(r) = -\frac{\mu}{r}\left[1 - \sum_{l=2}^{\infty} J_l \left(\frac{R_\oplus}{r}\right)^l P_l(\sin\varphi)\right]$$

The term of $l = 1$ is zero if the center of gravity is taken at the origin.
Assuming up to $l = 2$, we have

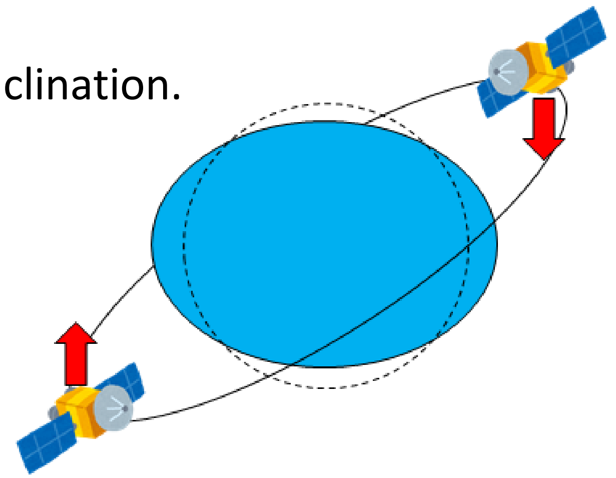
$$U(r) = -\frac{\mu}{r}\left[1 - J_2\left(\frac{R_\oplus}{r}\right)^2 P_2(\sin\varphi)\right], \quad P_2(\sin\varphi) = \frac{3(\sin\varphi)^2 - 1}{2}, \quad J_2 = -C_{20} = 1.08263 \times 10^{-3}$$

The term of $l = 2$ term is called the **J_2 term**, and the terms of $l \geq 3$ is quite small compared to the J_2 term.

J_2 term, cont.

The J_2 term represents the north-south distortion of the gravity field, which produces an action that moves the orbital plane closer to the equatorial plane.

Gyroscopic effect of this action and orbital motion perturbs RAAN, argument of perigee, and inclination.



$$\dot{\Omega} = -\frac{3}{2} n \frac{R_\oplus^2}{a^2(1-e^2)^2} J_2 \cos i \,, \quad n = \sqrt{\frac{\mu}{a^3}}$$

$$\dot{\omega} = -\frac{3}{4} n \frac{R_\oplus^2}{a^2(1-e^2)^2} J_2 (1 - 5\cos^2 i)$$

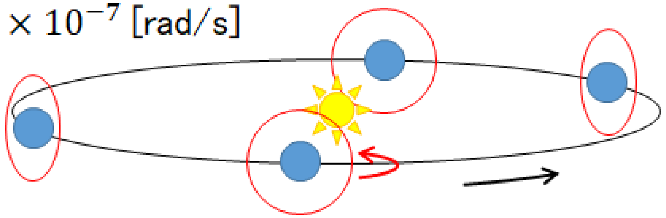
$$\dot{M} = n + \frac{3}{4} n \frac{R_\oplus^2}{a^2(1-e^2)^{3/2}} J_2 (3\cos^2 i - 1)$$

RAAN moves westward when the orbit inclination is less than 90 degrees,
and moves eastward when the orbit inclination is greater than 90 degrees.

1.2 Orbit Elements and Orbit Determination

- **Sun-Synchronous Orbit, SSO**

$\omega_{earth} = 1.991 \times 10^{-7}$ [rad/s]



By matching the change in RAAN due to the J_2 term to the angular velocity of the Earth's revolution, the angle between the orbital plane and the sun direction can be made nearly constant.

This makes it possible to maintain a constant amount of power generation throughout the year, and it is preferred for Earth, solar, and astronomical observation satellites because the positional relationship between the target and the sun as seen from the satellite is nearly constant.

Since the change in RAAN due to the J_2 term should match the angular velocity of the Earth revolution, the sun synchronization condition is

$$\dot{\Omega} = -\frac{3}{2} n \left(\frac{R_\oplus}{a}\right)^2 J_2 \cos i \quad \text{or} \quad -a^{7/2}(1-e^2)^2 = 2.0893 \times 10^{14} \cdot \cos i$$

From this, the relationship between the semimajor axis radius and the inclination can be obtained.

Example

Confirm that the orbit of Japanese Earth observation satellite "Daichi-2" satisfies the sun-synchronous condition.

Advanced Land Observing Satellite-2

H-IIA Launch Vehicle Flight 24, launching the Advanced Land Observing Satellite-2 "Daichi-2".

Names	Daichi-2
Mission type	Remote sensing
Operator	JAXA

Orbital parameters	
Reference system	Geocentric orbit[2]
Regime	Sun-synchronous orbit
Perigee altitude	636 km (395 mi)
Apogee altitude	639 km (397 mi)
Inclination	97.92°
Period	97.33 minutes

Advanced Land Observation Satellite

← ALOS ALOS-3 →

By https://en.wikipedia.org/wiki/ALOS-2 Adapted.

- **Earth Recurrent Orbit / Earth Sub-recurrent Orbit**

Earth recurrent orbit is an orbit in which a satellite orbits the Earth N times during one rotation of the Earth.
N is an integer and called the recurrent number of satellite.

Earth Sub-recurrent orbit is an orbit in which a satellite orbits the Earth N times during M times rotation of the Earth.
M is an integer and called the recurrent days of satellite.

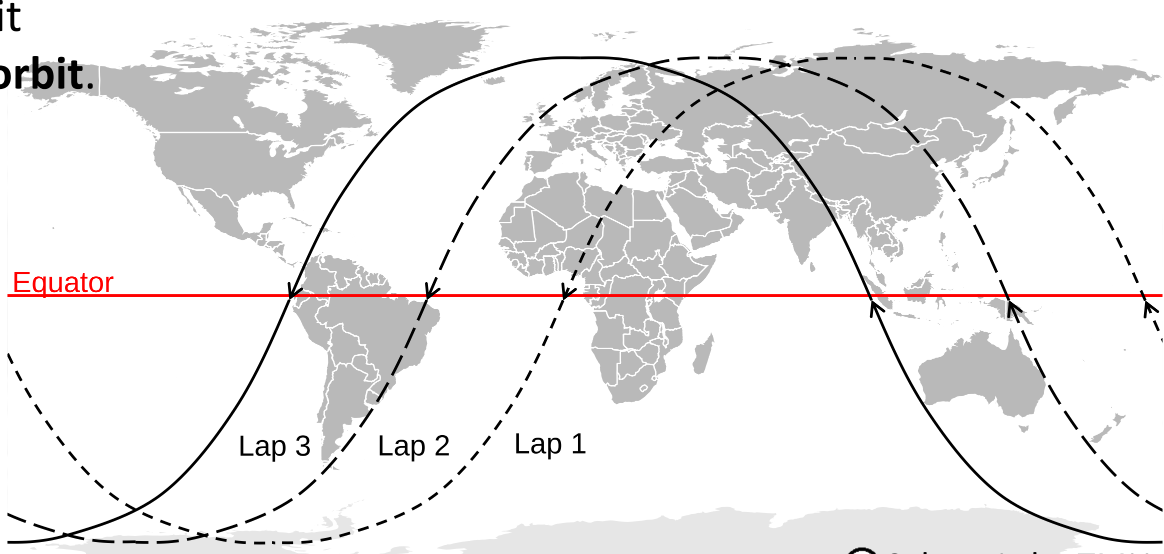
The recurrent/sub-recurrent condition is $\left(\omega_{\oplus} - \dot{\Omega}\right)N = (n + \dot{\omega})M$.
These reproduce the positional relation between the satellite and the Earth's surface at regular intervals.

Taking both conditions of sun-synchronous and (sub-)recurrent orbit into consideration realizes **sun-synchronous Earth (sub-)recurrent orbit**.

- **Geosynchronous Orbit**

The orbits with $N = 1$ and $M = 1$ (one sidereal day).

Geo-Stationary Orbit (GEO) is one of them, and is often used for permanent communication and weather observation because it always appears to be in the same position from the ground.



- **Molniya Orbit**

  It is difficult to launch geostationary satellites at the equator in high latitude countries, and the low elevation angle makes their operation inefficient. Therefore, an orbit with the following orbital elements was devised to allow a longer operational time by placing the apogee above the country:
  $a = 26{,}600[\mathrm{km}] \Longrightarrow T \simeq 12[\mathrm{hrs}]$, $e = 0.75$, and $i = 63.435°$ for $\dot{\omega} \simeq 0$.

https://en.wikipedia.org/wiki/Molniya_orbit

- **Tundra Orbit**

  For 24-hour operations in Europe, Tundra orbit with the following orbital elements requires only three satellites, compared to four units for a Molniya orbit:
  $r_p = 24{,}000[\mathrm{km}]$, $r_a = 47{,}000[\mathrm{km}] \Longrightarrow T \simeq 24[\mathrm{hrs}]$, and $i = 63.435°$ for $\dot{\omega} \simeq 0$.

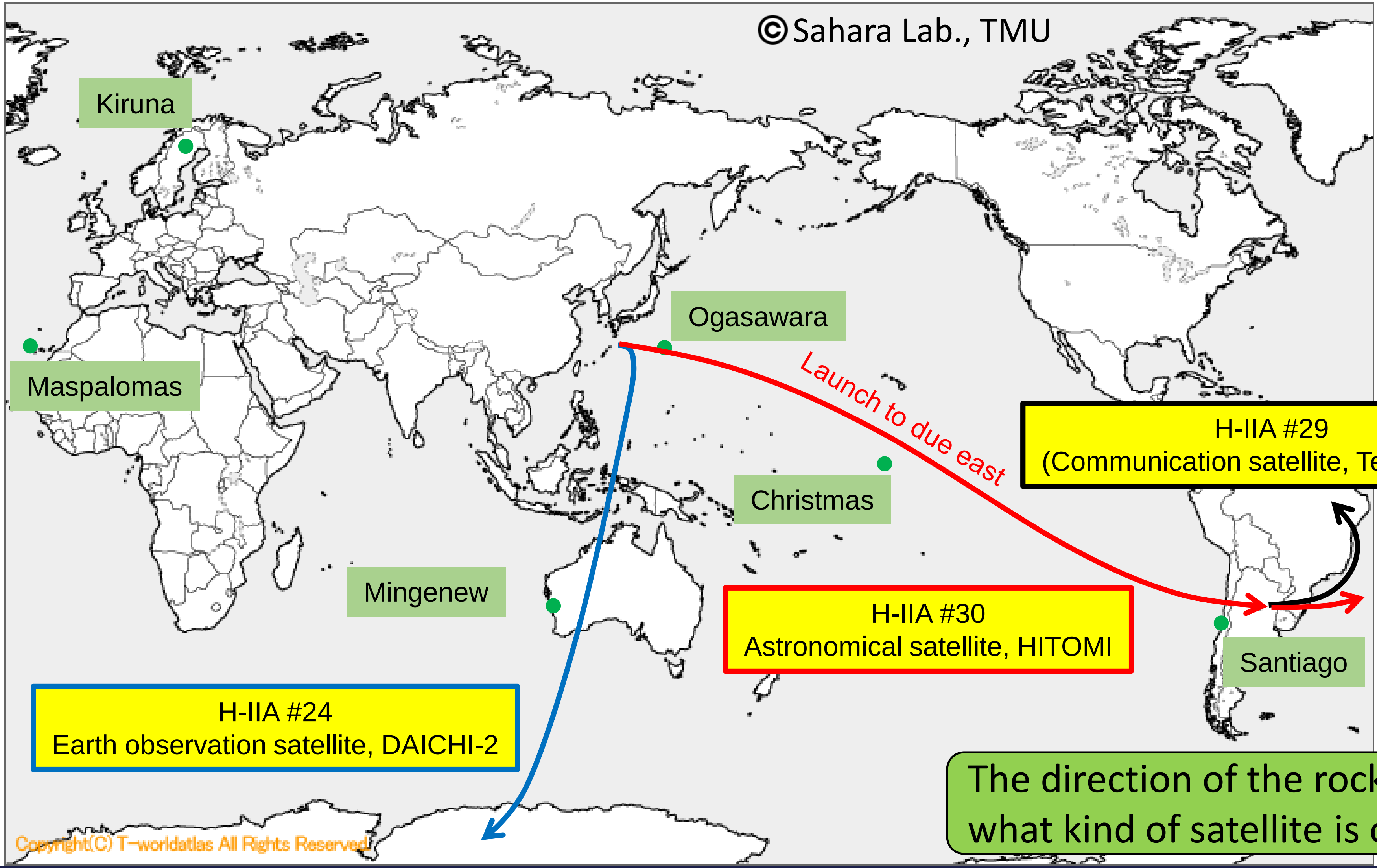
- **Quasi-Zenith Orbit, QZO**

  The orbit period is one sidereal day (geosynchronous orbit), and has an appropriate eccentricity and inclination so that the satellite can stay over a specific area for a long period of time.

  A satellite in a QZO is called a quasi-zenith satellite (QZS), and a constellation in a QZO is called a quasi-zenith satellite system (QZSS).
  Example: Japanese Michibiki, and so on.

"Quasi-Zenith Satellite System Orbit, which is an Inclined Geosynchronous Orbit (IGSO). Inclination: 45°; eccentricity: 0.09; argument of periapsis: 270°." © Tubas (Licensed under CC BY-SA 3.0)
https://ja.wikipedia.org/wiki/%E6%BA%96%E5%A4%A9%E9%A0%82%E8%A1%9B%E6%98%9F#/media/%E3%83%95%E3%82%A1%E3%82%A4%E3%83%AB:Qzss-45-0.09.jpg

1.2 Orbit Elements and Orbit Determination



The direction of the rocket's flight reveals what kind of satellite is on board.

Flight Time

The time of flight from perigee (P) to any point on the orbit (Q), expressed in eccentric anomaly, is obtained using Kepler's second law

$$\frac{t_Q - t_P}{S_{QFP}} = \frac{T}{\pi ab}, \quad T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

an elliptical orbit, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and auxiliary circle, $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$, as follows:

$$y_Q = \frac{b}{a}\sqrt{a^2 - x^2}, \quad y_B = \sqrt{a^2 - x^2} \quad \Longrightarrow \quad \frac{y_Q}{y_B} = \frac{b}{a} = \sqrt{1 - e^2}$$

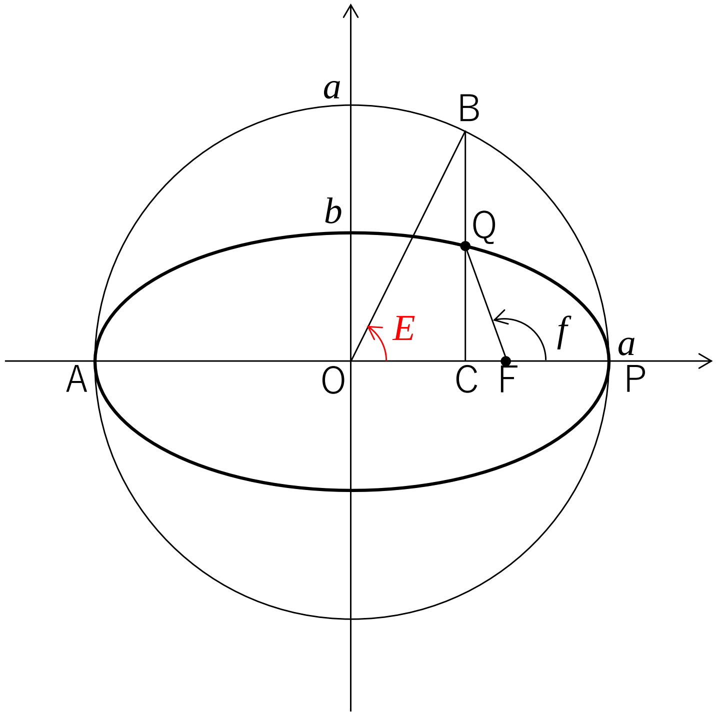
Then,

$$S_{QFP} = S_{QCP} - S_{QCF} = \frac{ab}{2}(E - e\sin E)$$

$$S_{QCF} = \frac{1}{2} \cdot (ae - a\cos E) \cdot \frac{b}{a} a \sin E = \frac{ab}{2}(e \sin E - \cos E \sin E)$$

$$S_{QCP} = \frac{b}{a} S_{BCP} = \frac{b}{a}(S_{OBP} - S_{OBC}) = \frac{b}{a}\left(\frac{a^2}{2}E - \frac{a^2}{2}\cos E \sin E\right) = \frac{ab}{2}(E - \cos E \sin E)$$

We obtain $\therefore t_Q - t_P = \frac{T}{\pi ab} S_{QFP} = \sqrt{\frac{a^3}{\mu}}(E - e \sin E)$



1. Basic of Orbital Mechanics

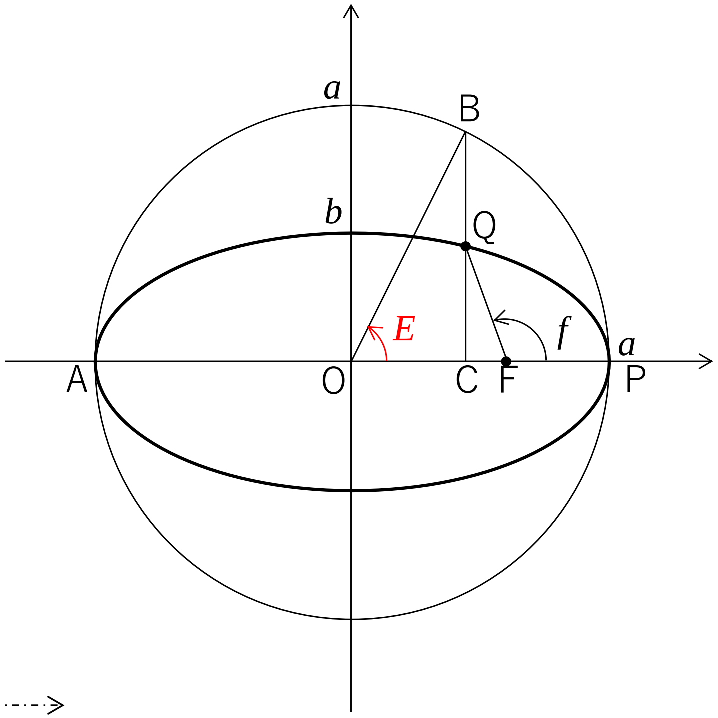
1.2 Orbit Elements and Orbit Determination

From the geometric relationship, $a\cos E = ae + r\cos f$

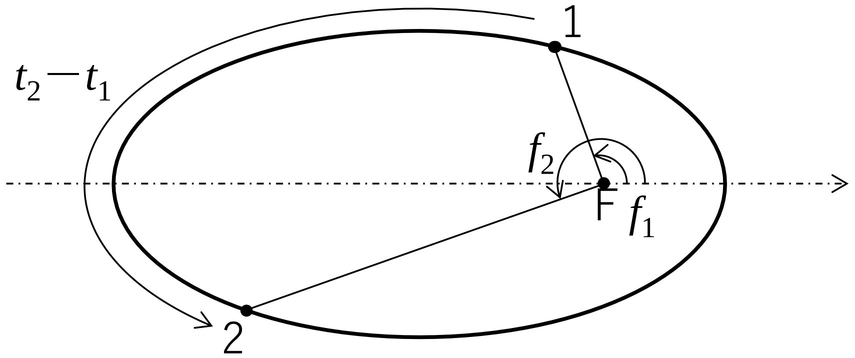
Also, $\cos E = \dfrac{ae+\frac{a(1-e^2)}{1+e\cos f}\cos f}{a} = \dfrac{e+\cos f}{1+e\cos f}$ from $r = \dfrac{p}{1+e\cos f} = \dfrac{a(1-e^2)}{1+e\cos f}$

That is, if e and f are known, E is obtained.

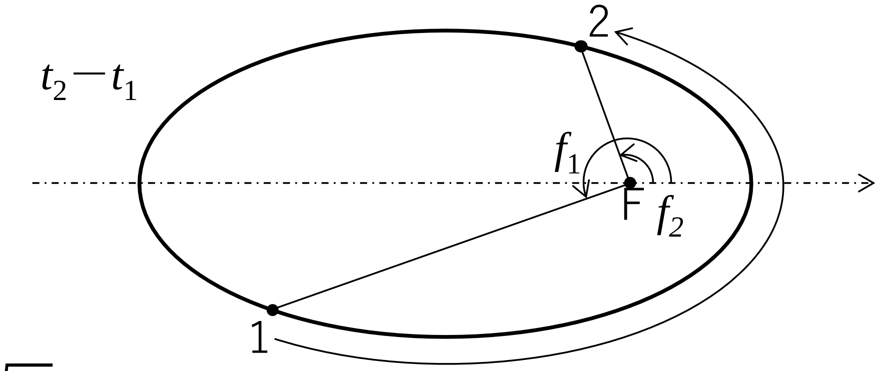
The time of flight between any two points is obtained as follows for i) without a perigee passage and ii) with a perigee passage.



i) $n = 0$



ii) n : Number of times passed perigee



$$t_2 - t_1 = nT + (t_2 - t_P) - (t_1 - t_P) = \sqrt{\frac{a^3}{\mu}}\left[2n\pi + (E_2 - e\sin E_2) - (E_1 - e\sin E_1)\right]$$

The following is for reference only.

Parabolic orbit

$t - t_p = \frac{1}{2\sqrt{\mu}}\left(pD + \frac{1}{3}D^3\right)$, where $D = \sqrt{p}\tan\frac{f}{2}$ is the eccentric anomaly in the case of parabolic orbit.

Hyperbolic orbit

The right-angled hyperbola (asymptotic lines are orthogonal) passing through the perigee is used as the auxiliary line.

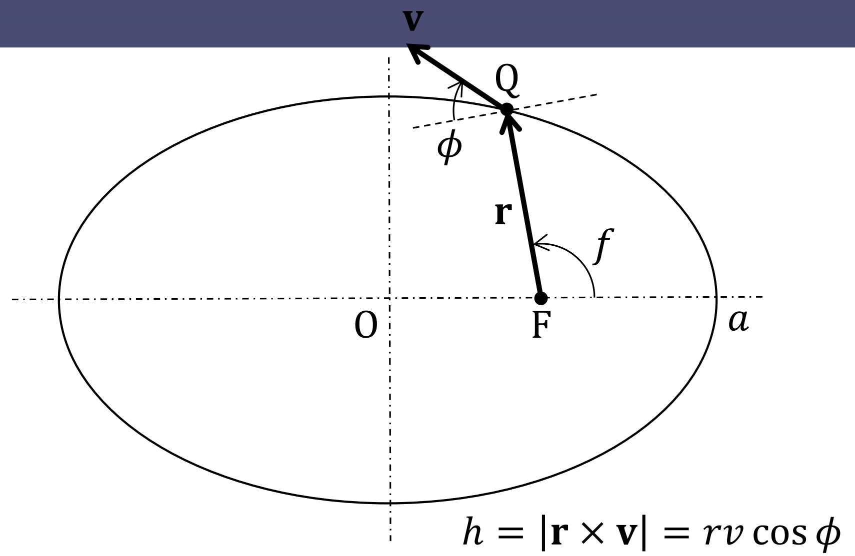
$$t - t_p = \sqrt{\frac{(-a)^3}{\mu}}(e\sinh F - F)$$

where $F = \ln\left(y + \sqrt{y^2 - 1}\right)$, $y = \cosh F = \frac{e+\cos f}{1+e\cos f}$ is the eccentric anomaly in the case of hyperbolic orbit.

$$F > 0 \ \ (0 \le f < \pi), \quad F < 0 \ \ (\pi \le f < 2\pi)$$

Single impulse orbit transfer

Orbital velocity is obtained as $E = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \implies v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$



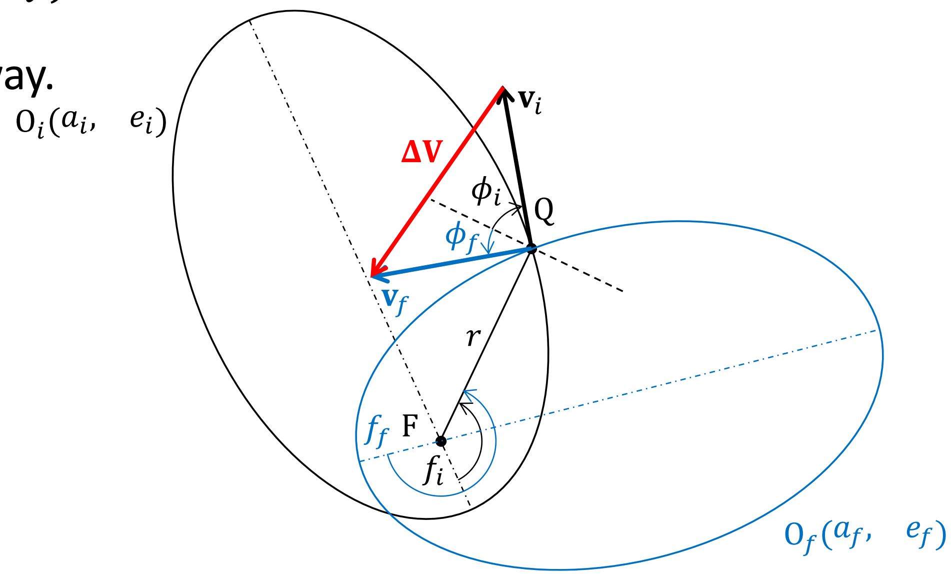
$h = |\mathbf{r} \times \mathbf{v}| = rv\cos\phi$

Supposed **the impulse approximation**, orbit transfer is completed instantaneously at the Q point from O_i to the coplanar orbit O_f.

The orbital velocity and flight path angle at Q on O_i are

$$v_i = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a_i}\right)} \quad \text{and} \quad \phi_i = \cos^{-1}\frac{h_i}{rv_i} \ \text{from} \ h_i = \sqrt{\mu p_i} = \sqrt{\mu a_i(1 - e_i^2)}$$

The orbital velocity and flight path angle at Q on O_f are obtained in the same way.



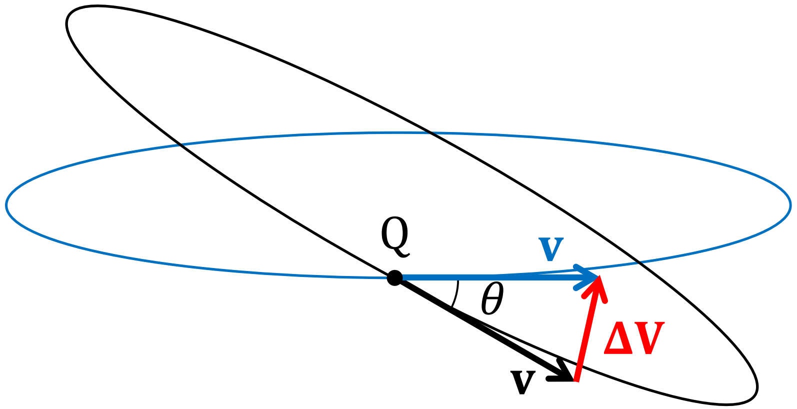
Therefore,
the velocity increment to be given is obtained using vector triangle and the cosine theorem as follows.

$$\Delta v = \sqrt{v_i^2 + v_f^2 - 2v_iv_f\cos(\phi_i + \phi_f)}$$

Single impulse orbit and plane transfer

In the case of orbit plane change, we obtain the velocity increment from $v_i = v_f = v$, as

$$\Delta V = \sqrt{v^2 + v^2 - 2v_i v_f \cos\theta} = \sqrt{2v^2(1-\cos\theta)} = 2v\sin\frac{\theta}{2}$$



These results indicate that transferring the orbit plane is quite difficult.

Therefore, orbit plane transfers should be conducted either by a rocket or at apogee, where orbital velocity is lower.

Note that **the velocity increment, $\boldsymbol{\Delta V}$**, is a **positive** value when calculating the propellant required as follows, even for deceleration.

Estimating ΔV is one of the major objectives of orbit design.

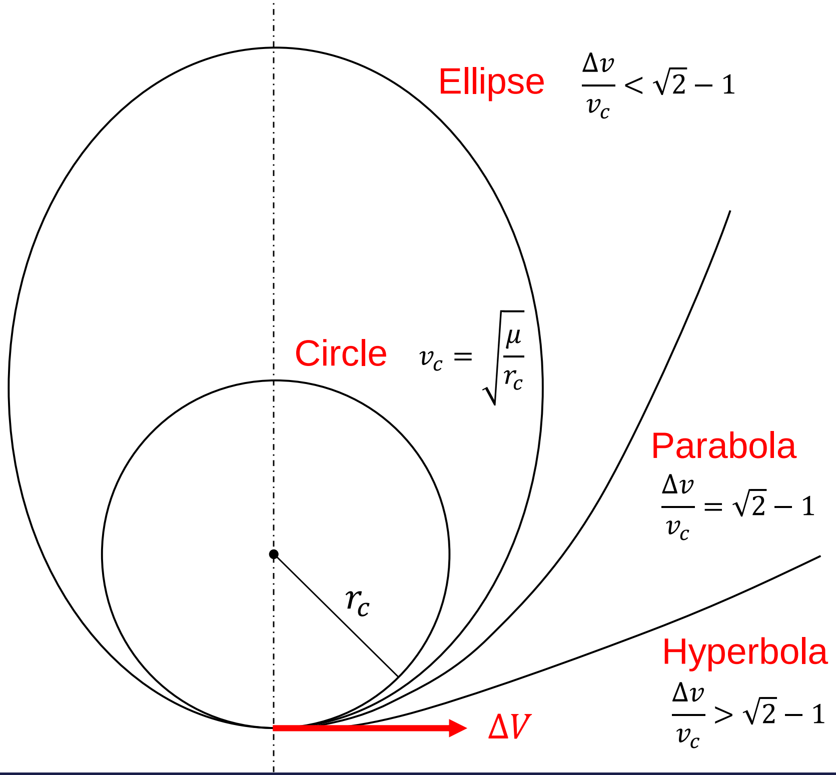
ΔV is used to judge mission feasibility, and also for satellite system design through **the Tsiolkovsky rocket equation**.

$$\Delta V = gI_{SP}\ln\frac{m_i}{m_f}$$

1.3 Orbit Transfer

If the velocity increment is given tangentially at a point on the circular orbit,

- The altitude of the injection point does not change.
- The altitude of the antipodal point of the injection point changes.
- As a result, the injection point is perigee and its antipodal point is apogee or infinity.



Hohmann transfer

A transfer from the initial orbit O_i via the transition orbit O_t to the target orbit O_f with two impulse injections.

1. Giving ΔV_1 is given at P in O_i,
   the antipodal altitude of P increases and moves to O_t.
2. Giving ΔV_2 is given at A in O_t,
   the antipodal altitude of P increases and moves to O_f.

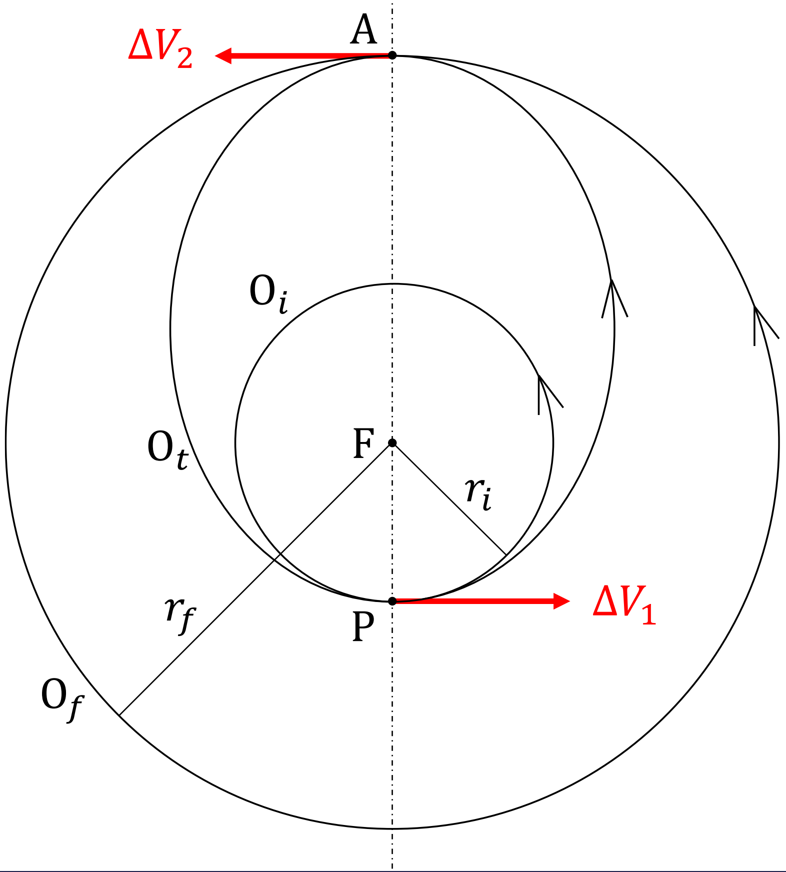
$a_t = \frac{r_f + r_i}{2}$ and $e_t = \frac{r_f - r_i}{r_f + r_i}$ for O_t,

$$v_\mathrm{P} = \sqrt{\mu\left(\frac{2}{r_i} - \frac{1}{a_t}\right)} = \sqrt{\frac{\mu}{r_i} \cdot \frac{2r_f}{r_f + r_i}} = v_i\sqrt{\frac{2r_f}{r_f + r_i}} \implies \Delta V_1 = v_\mathrm{P} - v_i = v_i\left[\sqrt{\frac{2r_f}{r_f + r_i}} - 1\right]$$

$$v_\mathrm{A} = \sqrt{\mu\left(\frac{2}{r_f} - \frac{1}{a_t}\right)} = \sqrt{\frac{\mu}{r_f} \cdot \frac{2r_i}{r_f + r_i}} = v_f\sqrt{\frac{2r_i}{r_f + r_i}} \implies \Delta V_2 = v_f - v_\mathrm{A} = v_f\left[1 - \sqrt{\frac{2r_i}{r_f + r_i}}\right]$$

Therefore, $\Delta v_{total} = \Delta v_1 + \Delta v_2$

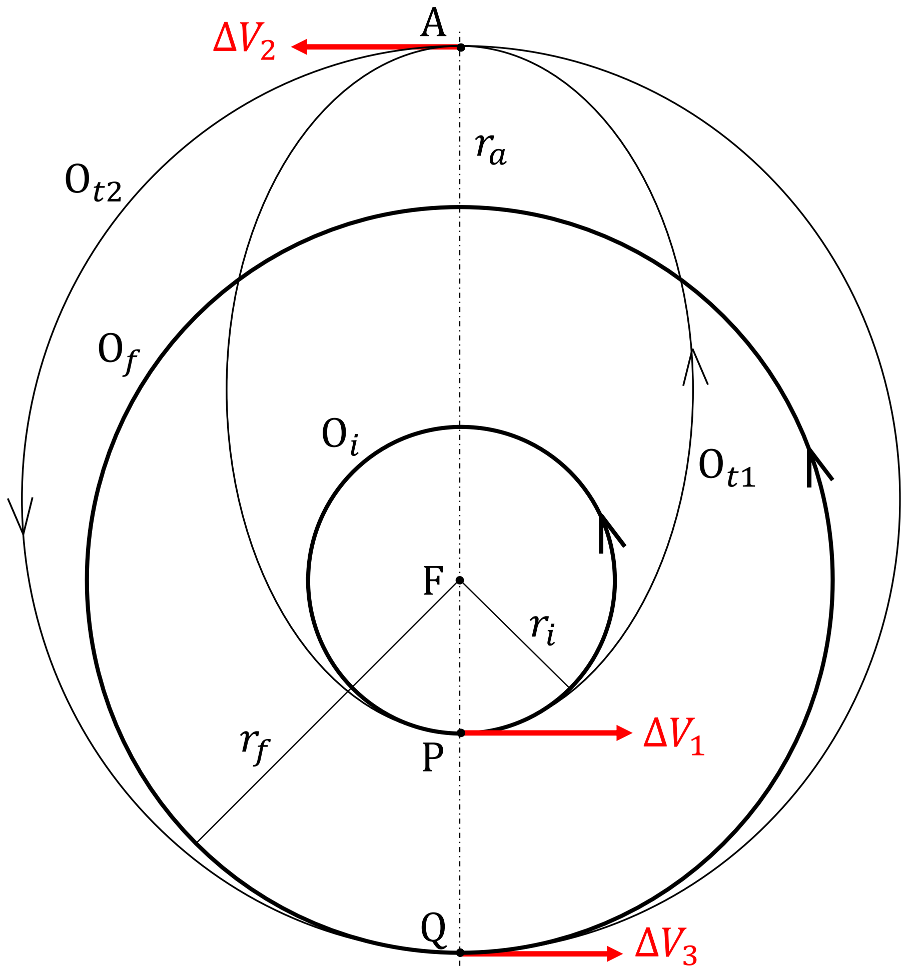
In the range $r_f / r_i < 11.9$, the minimum energy.



Bi-elliptic transfer

Three impulse orbit transfer via two transition orbits, Ot1 and Ot2.

By adjusting the altitude of A, the phase adjustment on the target orbit or the meeting time with the target on the target orbit can be adjusted.

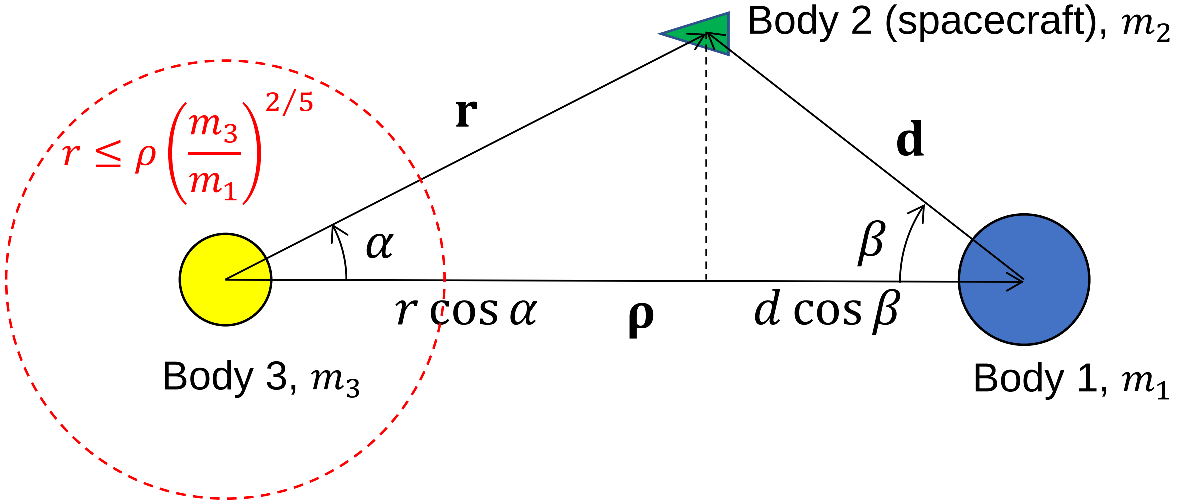


The following assumptions are made.

1. Planetary orbits are all in the ecliptic plane (i.e., co-planar orbit) and form a circular orbit around the sun.
2. During flight, they are subject only to the gravitational pull of the sun.

Considering the universal gravitation from Body 1 to Body 2 as principal and the universal gravitation from Body 3 as perturbation, it follows that the perturbation force on Body 2 from Body 3 is more dominant within the following distance range from the boundary.

$$r \approx \rho \left(\frac{m_3}{m_1}\right)^{2/5}$$



This range is called **the sphere of influence**.

Within the sphere of influence, the problem can be approximated as a two-body problem with Body 3 and Body 2, and outside the sphere of influence, with Body 1 and Body 2.

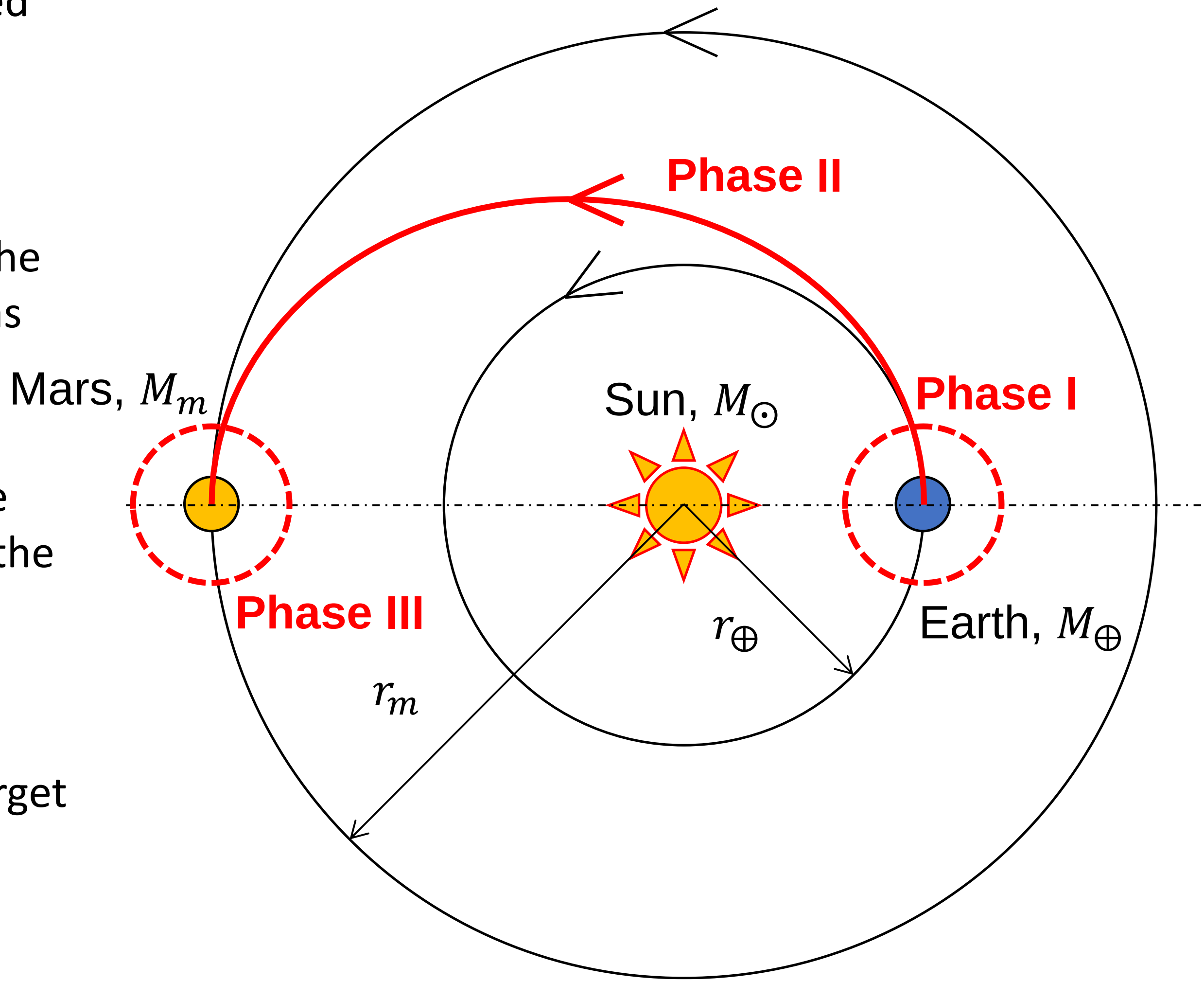
In the Sun-Earth system, the Earth's sphere of influence is

$$r \leq \rho \left(\frac{m_\oplus}{m_\odot}\right)^{2/5} = 1.496 \times 10^8 \cdot \left(\frac{5.974\times10^{24}}{1.989\times10^{30}}\right)^{0.4} \simeq 9.247 \times 10^5 \text{km}$$

1.4 Flight to the Moon and the Planets

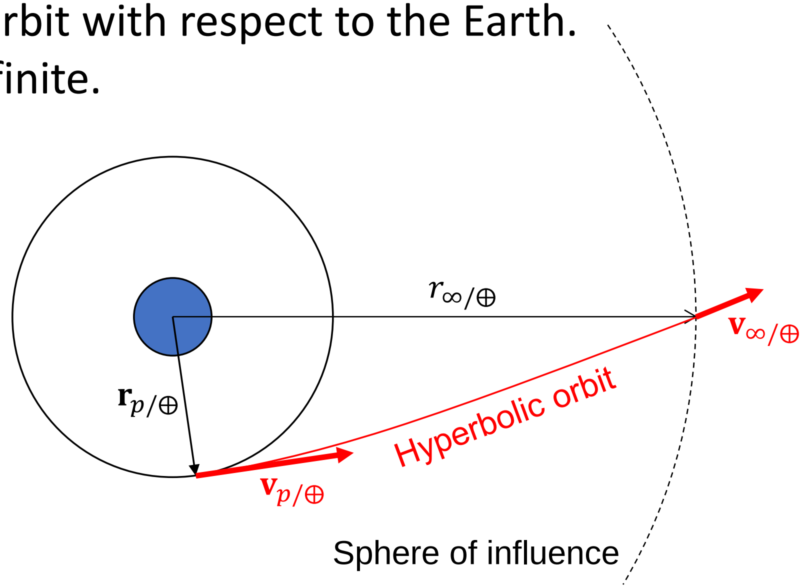
Considering the sphere of influence, interplanetary flight is divided into the following three phases.

1. **Phase I**: Departure Phase
   For the two-body problem of the Earth and the spacecraft, the spacecraft is inserted into a hyperbolic orbit with the Earth as the focal point.
2. **Phase II**: Interplanetary phase
   For the two-body problem of the sun and the spacecraft, the spacecraft is connected to an elliptical orbit with the sun as the focal point.
3. **Phase III**: Arrival Phase
   The spacecraft enters into a hyperbolic orbit with the target planet as the focal point for the two-body problem of the target planet and the spacecraft.



Escape from the Sphere of Influence

In order to escape the Earth's gravitation sphere,
It must have sufficient velocity to reach infinity from the Earth and takes a hyperbolic orbit with respect to the Earth.
The boundary of the sphere of influence is so far away and is regarded as practically infinite.
That is, it needs to have a positive velocity at the boundary of the sphere of influence.



From the conservation of energy equation,

$$E = \frac{v_{p/\oplus}^2}{2} - \frac{\mu_\oplus}{r_{p/\oplus}} = \frac{v_{\infty/\oplus}^2}{2} - \frac{\mu_\oplus}{r_{\infty/\oplus}} \longrightarrow \frac{v_{\infty/\oplus}^2}{2} \quad \because r_{\infty/\oplus} \to \infty$$

$$\therefore v_{p/\oplus} = \sqrt{v_{\infty/\oplus}^2 + \frac{2\mu_\oplus}{r_{p/\oplus}}} \quad \Leftrightarrow \quad v_{\infty/\oplus} = \sqrt{v_{p/\oplus}^2 - \frac{2\mu_\oplus}{r_{p/\oplus}}}$$

The velocity increment required to escape the sphere of influence from orbit radius $r_{p/\oplus}$ is $\Delta V = v_{p/\oplus} - \sqrt{\frac{\mu_\oplus}{r_{p/\oplus}}}$.

True anomaly when reaching the boundary of the sphere of influence, $r_{\infty/\oplus}$, is

$$\cos f_{\infty/\oplus} = \frac{1}{e}\left(\frac{p}{r_{\infty/\oplus}} - 1\right) \longrightarrow -\frac{1}{e} \quad \Longrightarrow \quad f_{\infty/\oplus} = \cos^{-1}\left(-\frac{1}{e}\right)$$

$$e = \sqrt{1 + \frac{2Eh^2}{\mu^2}}$$

Passage of Planet

A spacecraft from interplanetary space passes through the sphere of influence of planet B. Assume that planet B is stationary.

Distance between asymptote line and the planet : $\Delta = -a\sqrt{e^2 - 1}$
Deflection angle with passage : δ
True anomaly on asymptotic line with passage : $f_{\infty/B}$

From the law of energy conservation,

$$v_{\infty/B}^{-} = v_{\infty/B}^{+} = v_{\infty/B}, \quad E = \frac{v_{\infty/B}^2}{2} = -\frac{\mu_B}{2a} \quad \Longrightarrow a = -\frac{\mu_B}{v_{\infty/B}^2}$$

And,

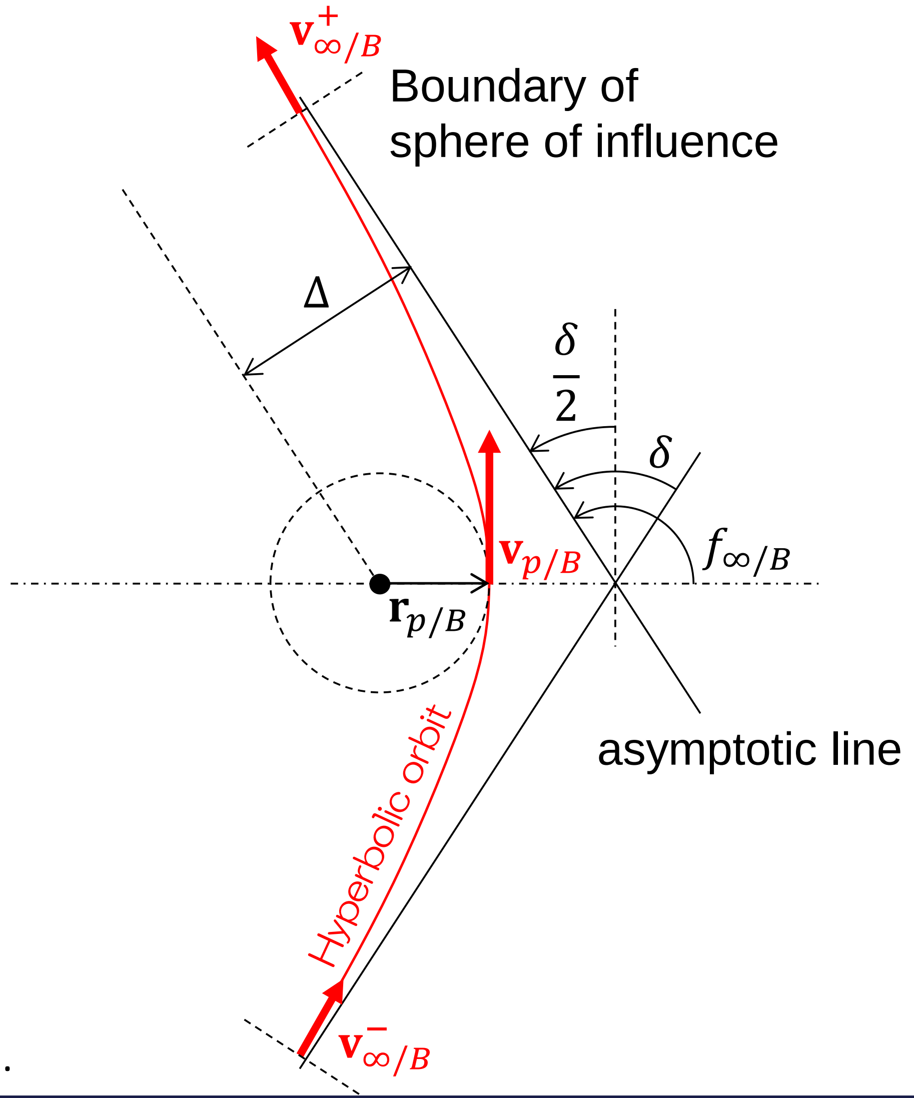
$$f_{\infty/B} = \cos^{-1}\left(-\frac{1}{e}\right) = \frac{\pi}{2} + \frac{\delta}{2} \quad \Longrightarrow \delta = 2\sin^{-1}\left(\frac{1}{e}\right)$$

Specific angular momentum is

$$h = v_{\infty/B}\Delta = \sqrt{\frac{\mu_B}{-a}} \cdot \left(-a\sqrt{e^2 - 1}\right) = \sqrt{\mu_B a(e^2 - 1)} = \sqrt{\frac{\mu_B^2}{v_{\infty/B}^2}(e^2 - 1)}$$

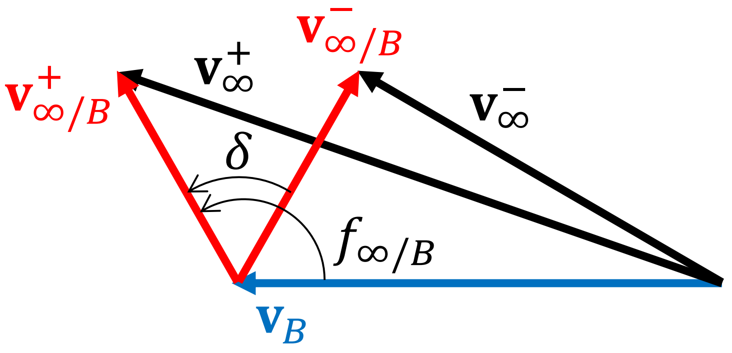
Eccentricity is $e^2 = 1 + \frac{v_{\infty/B}^4 \Delta^2}{\mu_B^2}$ or $e = 1 + \frac{r_{p/B} v_{\infty/B}^2}{\mu_B}$

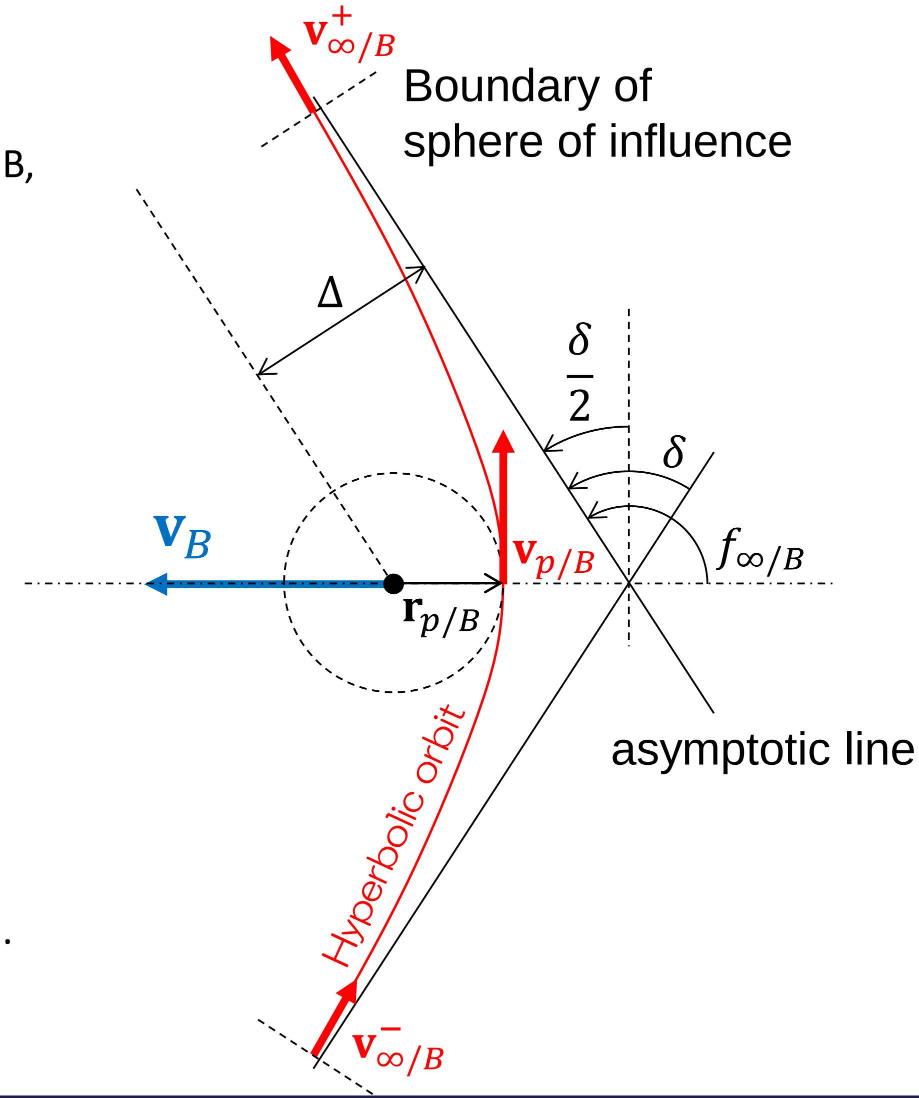
After all, Given $v_{\infty/B}$ and Δ, e and a are determined; $v_{p/B}$, $f_{\infty/B}$, and δ are also determined.



Swing-by, #1

Planet B is actually in motion at AA.
If we superimpose the velocities of planet B, the spacecraft from the perspective of planet B, and the spacecraft from the inertial coordinate system,





Can describe the relationship geometrically from velocity triangles.

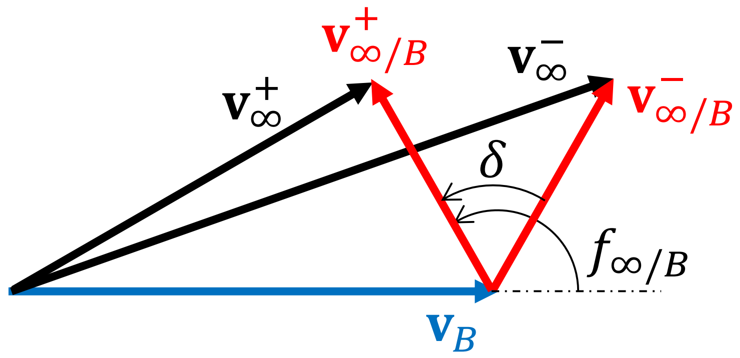
Before passing the planet : $v_{\infty}^{-} = \sqrt{v_B^2 + v_{\infty/B}^2 - 2v_B v_{\infty/B}\cos\left(f_{\infty/B} - \delta\right)}$

After passing the planet : $v_{\infty}^{+} = \sqrt{v_B^2 + v_{\infty/B}^2 - 2v_B v_{\infty/B}\cos f_{\infty/B}}$

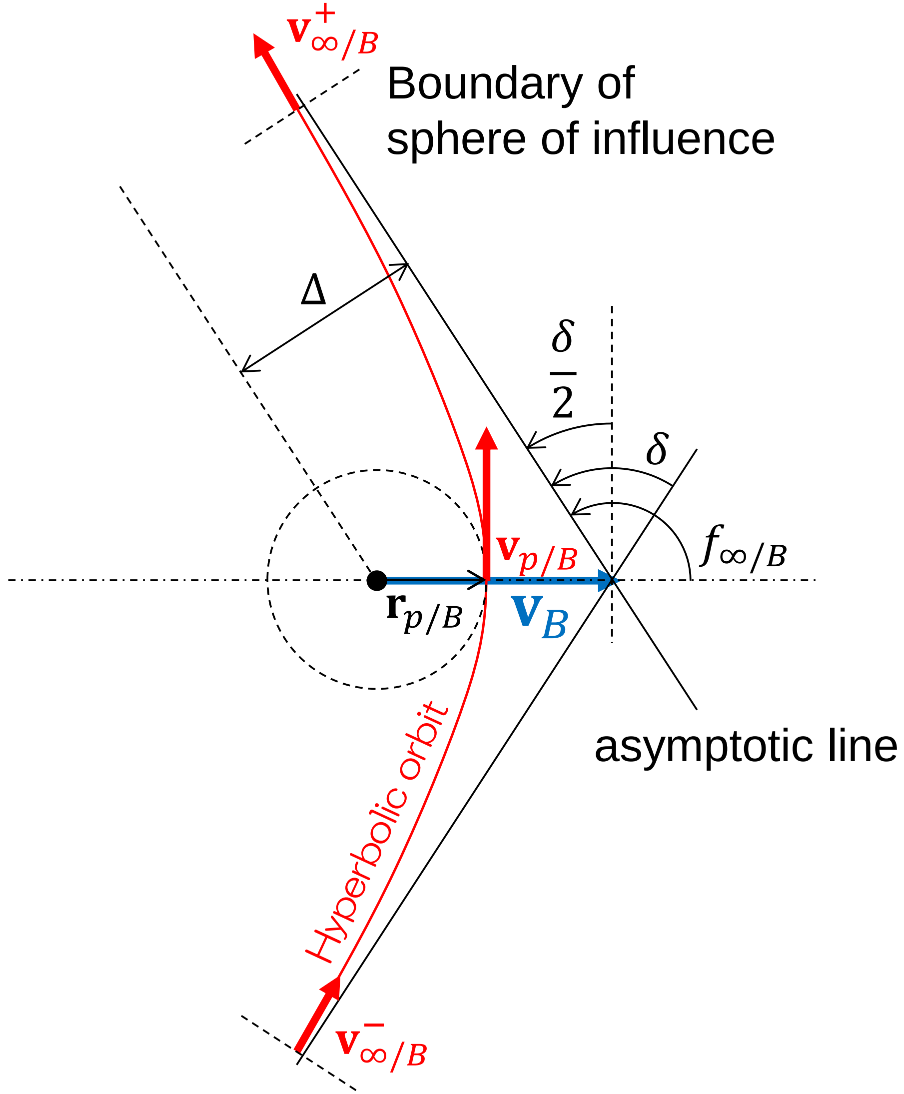
When $\cos f_{\infty/B} < \cos\left(f_{\infty/B} - \delta\right)$, the velocity is increased by passing the planet, $v_{\infty}^{+} > v_{\infty}^{-}$.
This is called **swing-by-acceleration**.

Swing-by, #1

Considering the case on the right figure in the same way, $v_\infty^+ < v_\infty^-$.
This is called **swing-by-deceleration**.



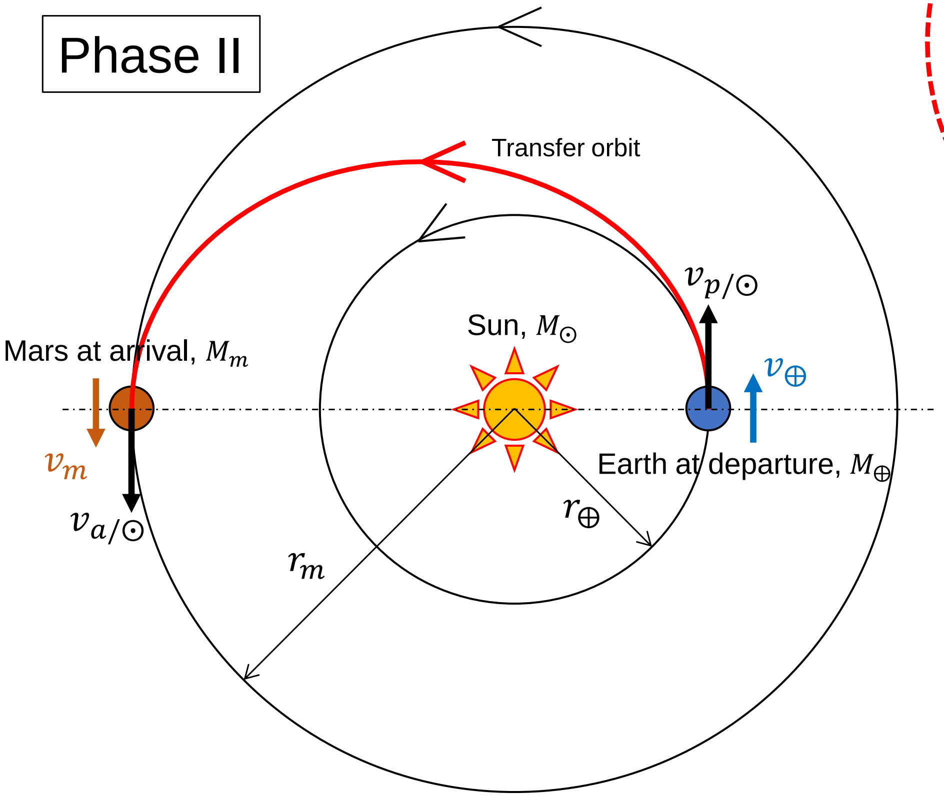
So, roughly speaking,
if it passes **behind** the planet, it is an swing-by-acceleration,
if it passes **in front of** the planet, it is a swing-by-deceleration.

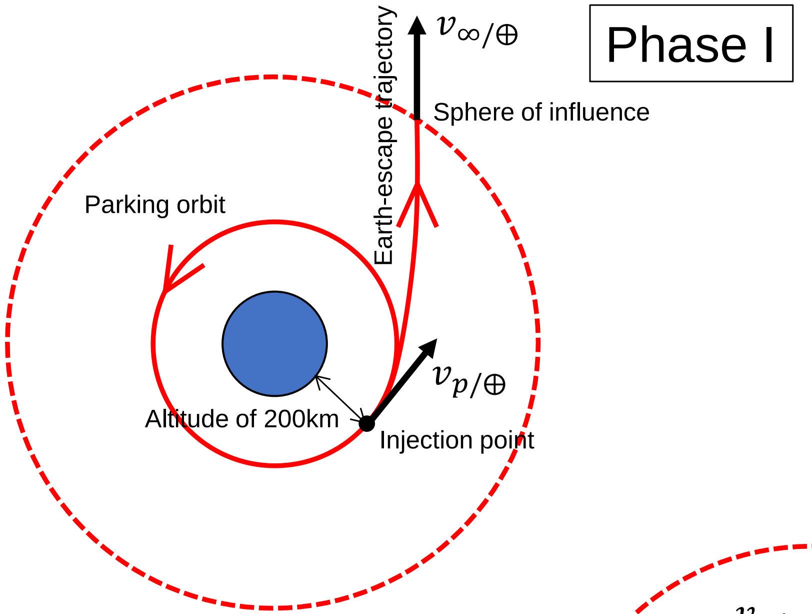


Flight to Mars

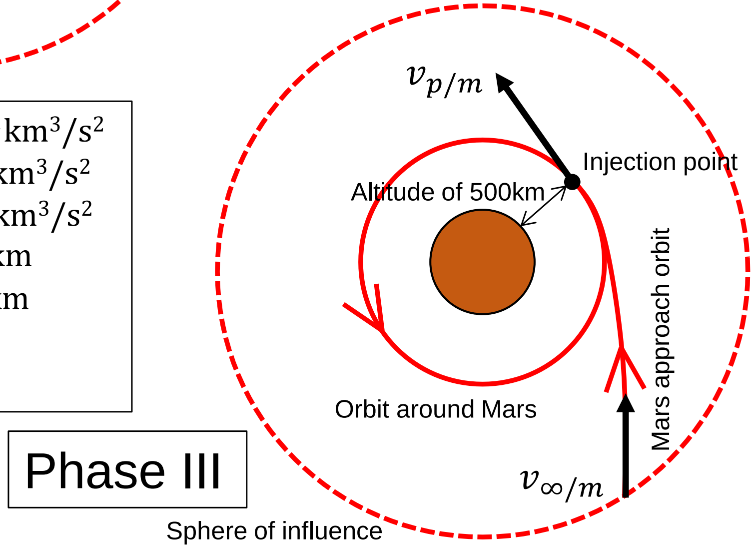
Initial study of the Mars Orbiter Program.

In general, it is easiest to start with Phase II.





$\mu_\odot = 1.327 \times 10^{11} \text{km}^3/\text{s}^2$
$\mu_\oplus = 3.986 \times 10^{5} \text{km}^3/\text{s}^2$
$\mu_m = 4.283 \times 10^{4} \text{km}^3/\text{s}^2$
$r_\oplus = 1.496 \times 10^{8} \text{km}$
$r_m = 2.279 \times 10^{8} \text{km}$
$R_\oplus = 6378 \text{km}$
$R_m = 3397 \text{km}$



Flight to Mars, Phase II

Semimajor axis of the transfer orbit : $a_H = \frac{r_\oplus + r_m}{2} = 1.8875 \times 10^8 \text{km}$

Velocity at perihelion : $v_{p/\odot} = \sqrt{\mu_\odot \left(\frac{2}{r_\oplus} - \frac{1}{a_H}\right)} = 32.73 \text{km/s}$

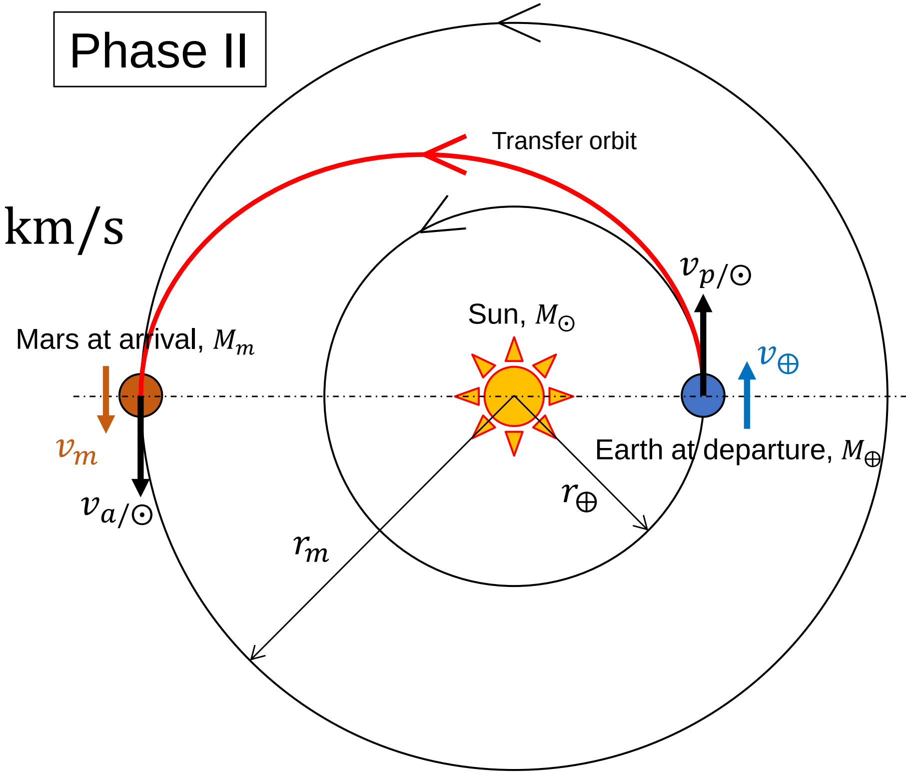
Velocity at escape from the Earth's sphere of influence:

$$v_\oplus = \sqrt{\frac{\mu_\odot}{r_\oplus}} = 29.78 \text{km/s}, \quad v_{\infty/\oplus} = v_{p/\odot} - v_\oplus = 2.95 \text{km/s}$$

Velocity at aphelion : $v_{a/\odot} = \sqrt{\mu_\odot \left(\frac{2}{r_m} - \frac{1}{a_H}\right)} = 21.48 \text{km/s}$

Velocity of entry into the sphere of influence of Mars:

$$v_m = \sqrt{\frac{\mu_\odot}{r_m}} = 24.13 \text{km/s}, \quad v_{\infty/m} = v_{a/\odot} - v_m = -2.65 \text{km/s}$$



Note carefully the direction of the arrows in the above figure (defining the direction of motion). This negative value means that the spacecraft will enter from the front of Mars.

Flight to Mars, Phase I

Phase I

Position at injection point : $r_{p/\oplus} = 200 + 6378 = 6578\text{km}$

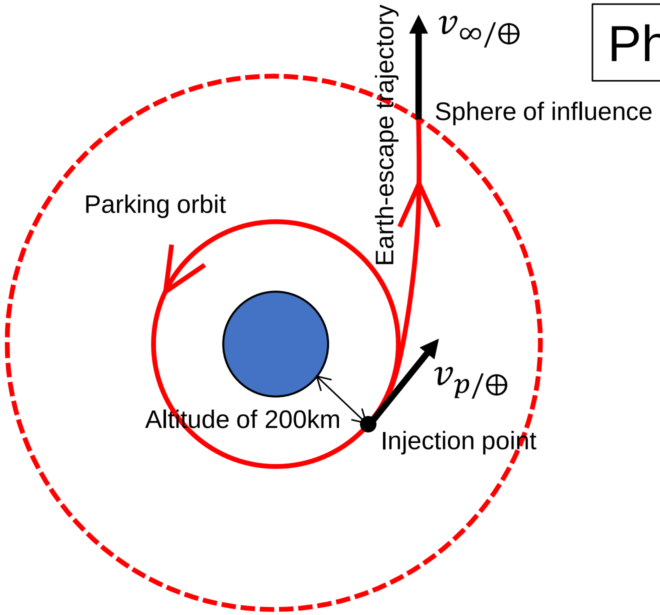
For the Earth-escape trajectory : $v_{\infty/\oplus} = 2.95\text{km/s}$

Patched-Conic

Then,

$$v_{p/\oplus} = \sqrt{v_{\infty/\oplus}^2 + \frac{2\mu_\oplus}{r_{p/\oplus}}} = \sqrt{2.95^2 + \frac{2 \cdot 3.986 \times 10^5}{200 + 6378}} = 11.40\text{km/s}$$

For the parking orbit : $v_{c/\oplus} = \sqrt{\frac{\mu_\oplus}{r_{c/\oplus}}} = \sqrt{\frac{3.986\times10^5}{200+6378}} = 7.78\text{km/s}$



Therefore, the velocity increment to be given at the injection point is

$$\Delta V_1 = v_{p/\oplus} - v_{c/\oplus} = \mathbf{3.62km/s}$$

The followings are obtained for the Earth-escape trajectory:

$$e = 1 + \frac{r_{p/\oplus} v_{\infty/\oplus}^2}{\mu_\oplus} = 1 + \frac{6578 \cdot 2.95^2}{3.986\times10^5} = 1.14, \quad f_{\infty/\oplus} = \cos^{-1}(-1/e) = 151.3°, \quad \frac{\delta}{2} = f_{\infty/\oplus} - \frac{\pi}{2} = 61.3°$$

$$\Delta = \frac{\mu_\oplus}{v_{\infty/\oplus}^2}\sqrt{e^2 - 1} = \frac{3.986 \times 10^5}{2.95^2}\sqrt{1.14^2 - 1} = 1.372 \times 10^4\text{km}$$

This is sufficiently small compared to interplanetary space. Therefore, it is reasonable to set the starting point in Phase II at the Earth's position.

Flight to Mars, Phase III

Position at injection point : $r_{p/m} = 500 + 3397 = 3897\text{km}$

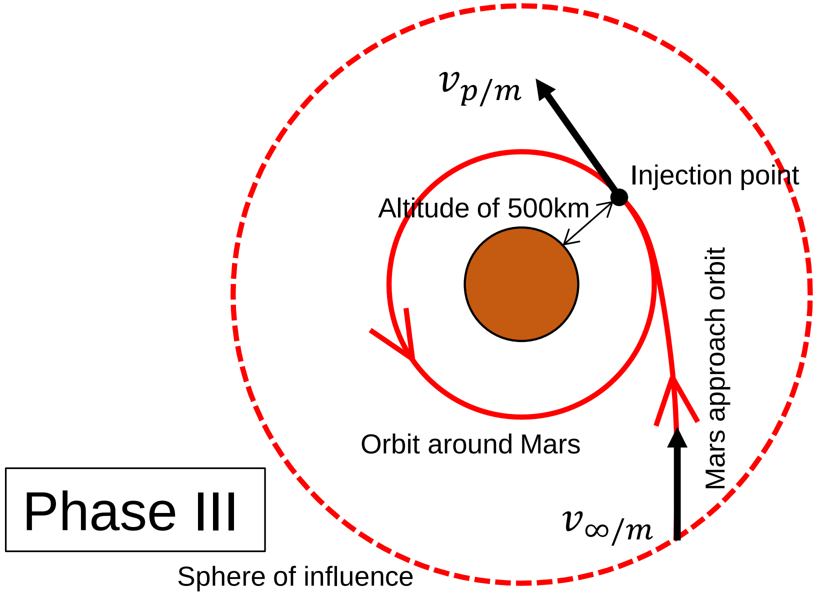
For the Mars approach trajectory : $v_{\infty/m} = -2.65\text{km/s}$

Patched-Conic

Then,

$$v_{p/m} = \sqrt{v_{\infty/m}^2 + \frac{2\mu_m}{r_{p/m}}} = \sqrt{2.65^2 + \frac{2 \cdot 4.283 \times 10^4}{500 + 3397}} = 5.39\text{km/s}$$

For the orbit around Mars : $v_{c/m} = \sqrt{\frac{\mu_m}{r_{c/m}}} = \sqrt{\frac{4.283\times10^4}{500+3397}} = 3.32\text{km/s}$



Therefore, the velocity increment to be given at the injection point is

$$\Delta V_2 = v_{c/m} - v_{p/m} = -2.07\text{km/s}$$

The followings are obtained for the Mars approach trajectory:

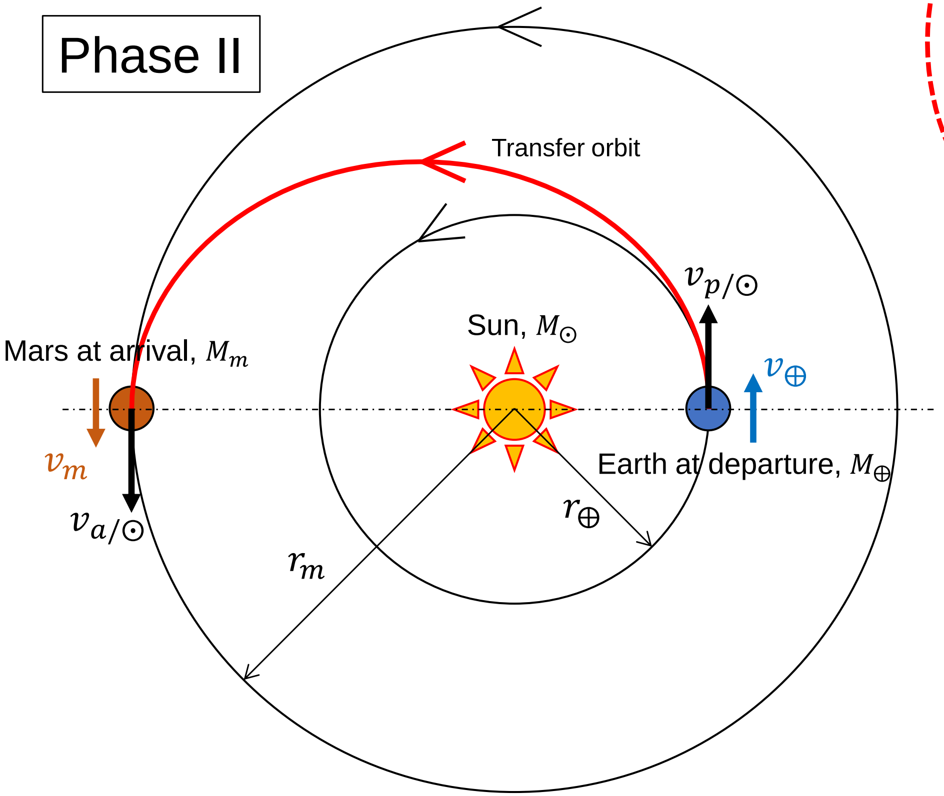
$$e = 1 + \frac{r_{p/m} v_{\infty/m}^2}{\mu_m} = 1 + \frac{3897 \cdot 2.65^2}{4.283\times10^4} = 1.65, \quad f_{\infty/m} = \cos^{-1}(-1/e) = 127.6°, \quad \frac{\delta}{2} = f_{\infty/m} - \frac{\pi}{2} = 37.6°$$

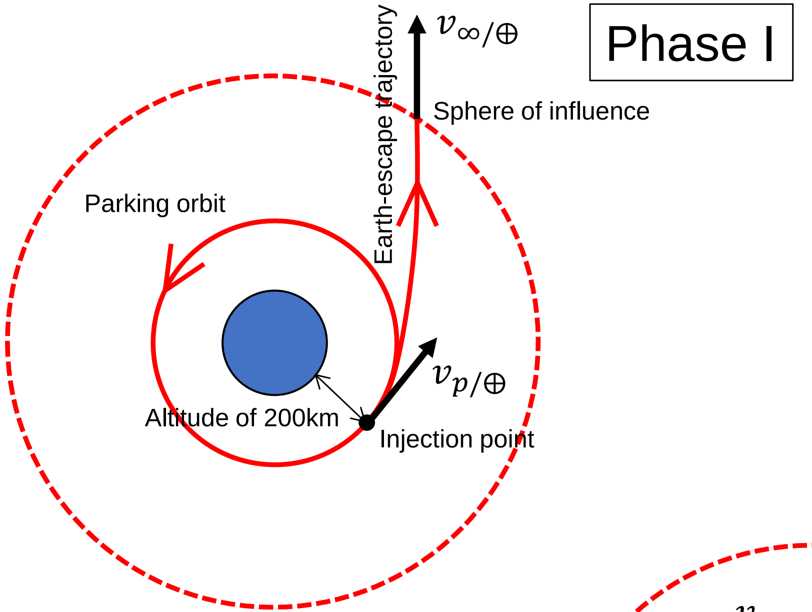
$$\Delta = \frac{\mu_m}{v_{\infty/m}^2}\sqrt{e^2 - 1} = \frac{4.283 \times 10^4}{2.65^2}\sqrt{1.64^2 - 1} = 7.928 \times 10^3\text{km}$$

This is sufficiently small compared to interplanetary space. Therefore, it is reasonable to set the arrival point in Phase II at the position of Mars.

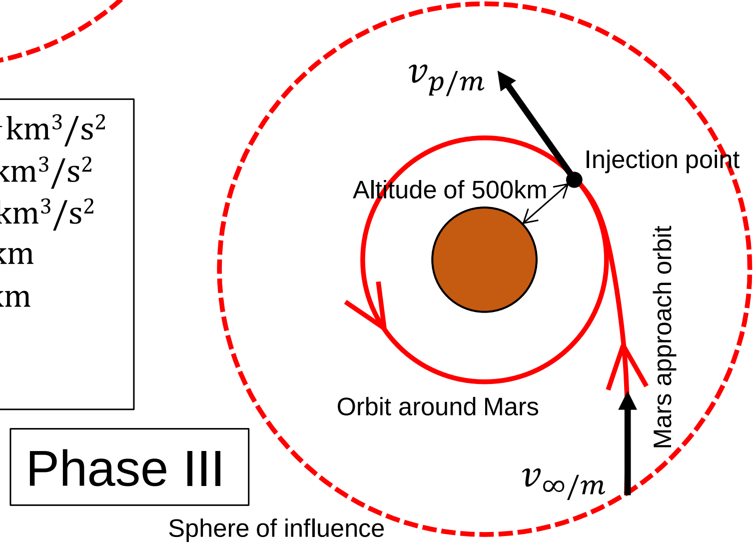
Flight to Mars, summary

From the above, it is concluded that the necessary velocity increment for the spacecraft in the Mars Orbiter Program is $\Delta V_{total} = \Delta V_1 + |\Delta V_2| = 3.62 + 2.07 = \mathbf{5.69 km/s}$.





$\mu_\odot = 1.327 \times 10^{11} \mathrm{km^3/s^2}$
$\mu_\oplus = 3.986 \times 10^{5} \mathrm{km^3/s^2}$
$\mu_m = 4.283 \times 10^{4} \mathrm{km^3/s^2}$
$r_\oplus = 1.496 \times 10^{8} \mathrm{km}$
$r_m = 2.279 \times 10^{8} \mathrm{km}$
$R_\oplus = 6378 \mathrm{km}$
$R_m = 3397 \mathrm{km}$



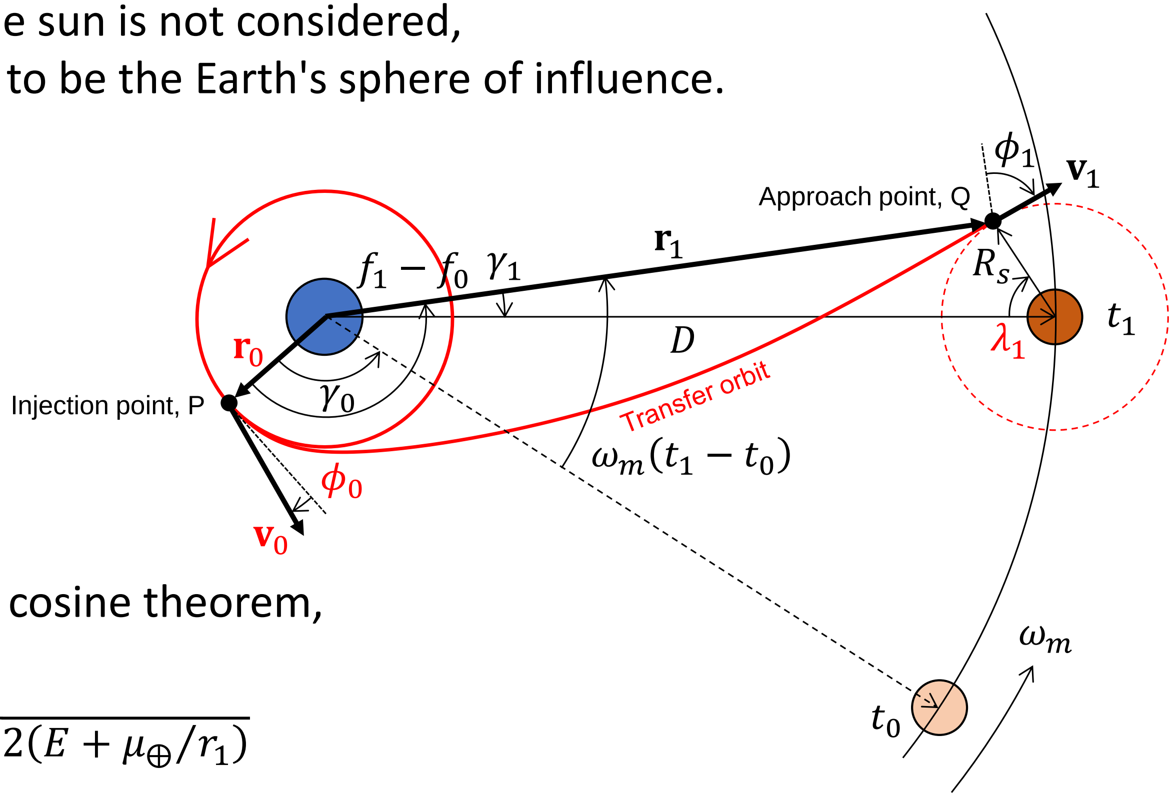
Flight to the Moon

Since the Earth and the moon are close, the sphere of influence of the sun is not considered, and the sphere outside the moon's sphere of influence is considered to be the Earth's sphere of influence.

Initial conditions : $\mathbf{r}_0, \quad \mathbf{v}_0, \quad \phi_0$
Termination condition: λ_1

For the injection point (P), Specific dynamic energy: $= \frac{v_0^2}{2} - \frac{\mu_\oplus}{r_0}$,

and specific angular momentum: $h = r_0 v_0 \cos\phi_0$



For the boundary of the lunar sphere of influence (point Q),

Let Rs be the range of the moon's sphere of influence, then from the cosine theorem, we have $r_1 = \sqrt{D^2 + R_s^2 - 2DR_s\cos\lambda_1}$

From the law of energy conservation, $E = v_1^2/2 - \mu_\oplus/r_1 \quad \longrightarrow \quad v_1 = \sqrt{2(E + \mu_\oplus/r_1)}$

From the law of conservation of angular momentum, $\phi_1 = \cos^{-1}\frac{h}{r_1 v_1}$

From the geometric relationship, $r_1\sin\gamma_1 = R_s\sin\lambda_1$

Departure to the boundary of the Moon's sphere of influence

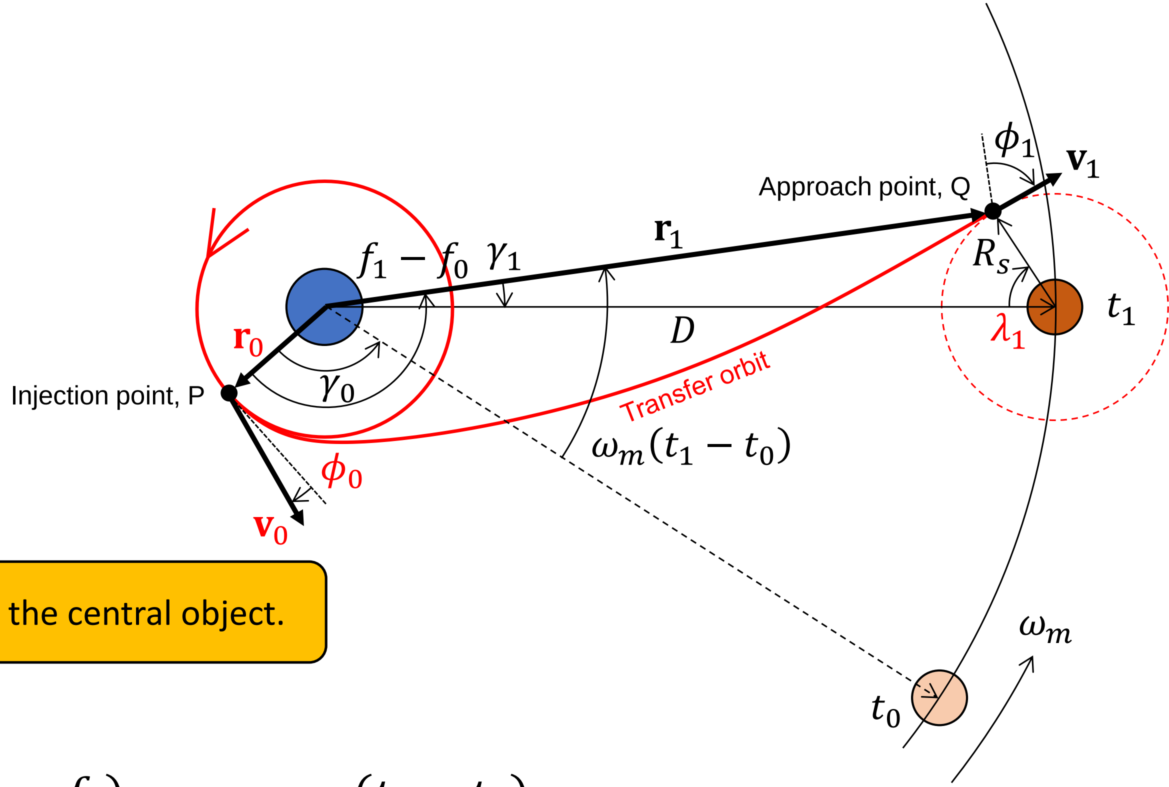
For the transfer orbit: $p = \frac{h^2}{\mu_\oplus}$, $a = -\frac{\mu_\oplus}{2E}$, $e = \sqrt{1 - \frac{p}{a}}$

Then, $f_i = \cos^{-1}\frac{p - r_i}{r_i e}$ from $r = \frac{p}{1 + e\cos f}$

and the eccentric anomaly: $E_i = \cos^{-1}\frac{e + \cos f_i}{1 + e\cos f_i}$

Therefore, $t_1 - t_0 = \sqrt{\frac{a^3}{\mu_\oplus}}\left[(E_1 - e\sin E_1) - (E_0 - e\sin E_0)\right]$

Note that so far this is the case with the Earth as the central object.



Phase condition

During $t_1 - t_0$, the moon orbits by $\omega_m(t_1 - t_0)$.

From the geometric relationship, the initial phase should be $\gamma_0 = (f_1 - f_0) - \gamma_1 - \omega_m(t_1 - t_0)$.

In practice, r_0, v_0, ϕ_0, and γ_0 are tried and tested in consideration of mission requirements

Connection condition

The initial value of the position of the moon-centered orbit is

$$r_2 = R_s$$

and the relative velocity of the spacecraft to the moon center is

$$\mathbf{v}_2 = \mathbf{v}_1 - \mathbf{v}_m, \quad v_m = D\omega_m$$

From the cosine theorem,

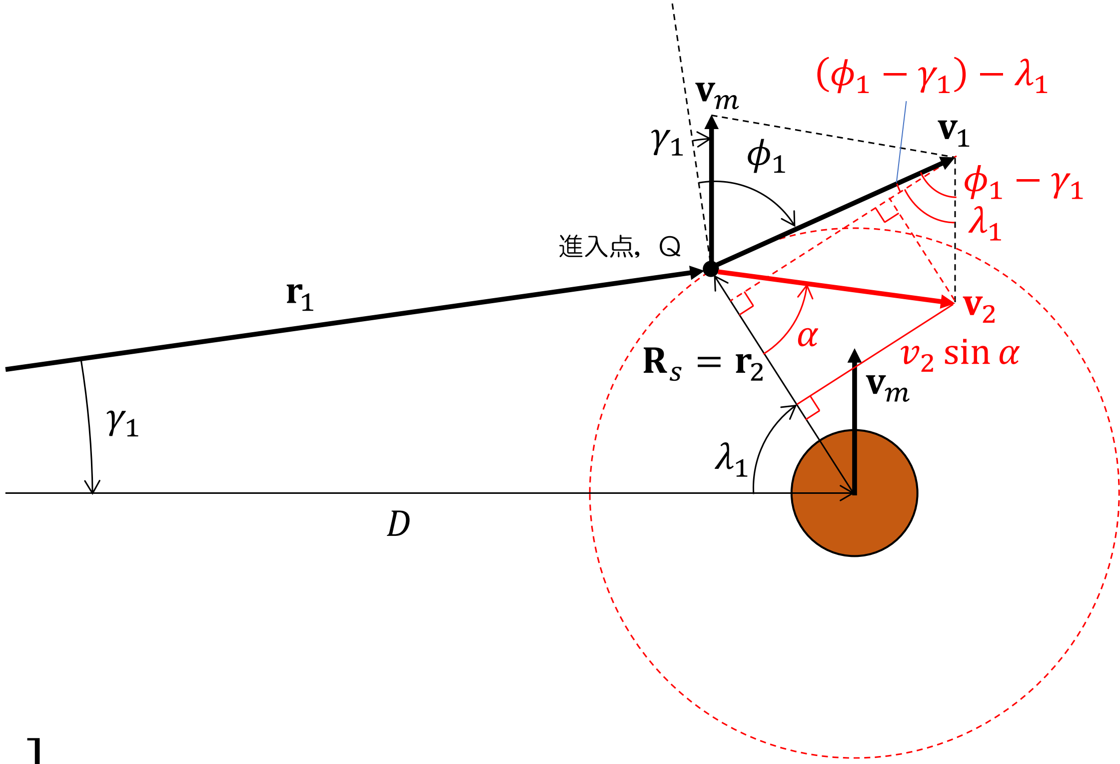
$$v_2 = \sqrt{v_1^2 + v_m^2 - 2v_1 v_m \cos(\phi_1 - \gamma_1)}$$

From geometric relations,

$$v_2 \sin\alpha = v_1 \cos[(\phi_1 - \gamma_1) - \lambda_1] - v_m \cos\lambda_1$$

$$\longrightarrow \quad \alpha = \sin^{-1}\left[\frac{v_1}{v_2}\cos[(\phi_1 - \gamma_1) - \lambda_1] - \frac{v_m}{v_2}\cos\lambda_1\right]$$

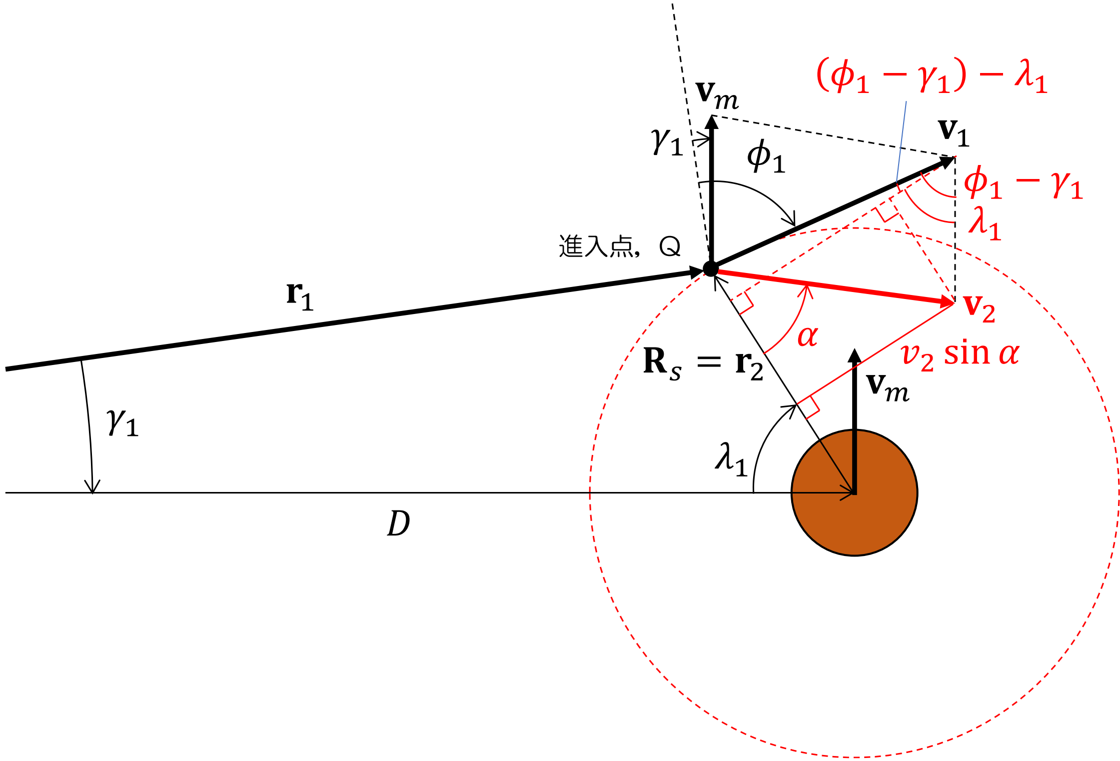
The above conditions, r_2, v_2, and α (position and velocity vectors from the moon's viewpoint at the time of moon entry) were obtained.



In the lunar perspective,

the specific dynamic energy is $E_m = \frac{v_2^2}{2} - \frac{\mu_m}{r_2}$,

and the specific angular momentum is $h_m = r_2 v_2 \sin\alpha$.



From the above, $p_m = \frac{h_m^2}{\mu_m} = a_m(1 - e_m^2)$, $e_m = \sqrt{1 + \frac{2E_m h_m^2}{\mu_m^2}}$

At the perilune, $r_{mp} = \frac{p_m}{1+e_m}$, $v_m = \sqrt{2\left(E_m + \frac{\mu_m}{r_{mp}}\right)}$

- When $r_{mp} > R_m$, do nothing and **flyby** the moon.
- When $r_{mp} < R_m$, impinges on the moon and becomes an **impactor**.
- Decelerates of $\Delta V_m = v_{mp} - \sqrt{\frac{\mu_m}{r_{mp}}}$ at perihelion, enters lunar orbit, and becomes an **orbiter**.
- Decelerates further, makes a soft landing, and becomes a **lander**.

2. Development of Orbital Mechanics

This chapter introduces some of the more advanced topics of orbital mechanics.

2.1 Planetary Equation

Gaussian Planetary Equations

is easier to use in orbit design than "**Lagrangian planetary equation**" because it can introduce perturbations and thrust.

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{n\sqrt{1-e^2}}\left(e \sin f\, F_r + \frac{p}{r} F_\theta\right), \quad n = \sqrt{\frac{\mu}{a^3}}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{\sqrt{1-e^2}}{na}\left[\sin f\, F_r + \left(\cos f + \frac{e+\cos f}{1+e\cos f}\right) F_\theta\right]$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{r\cos(\omega+f)}{na^2\sqrt{1-e^2}} F_z$$

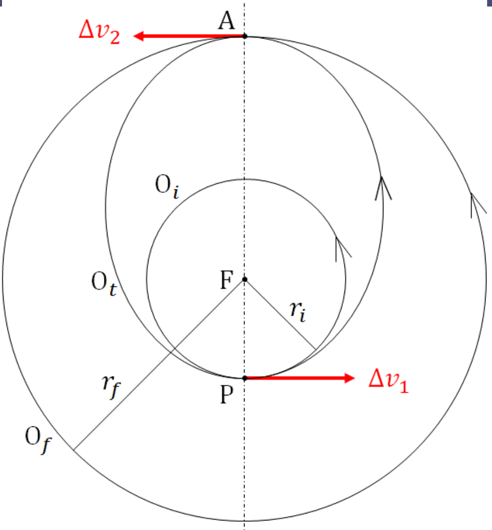
$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{r\sin(\omega+f)}{na^2\sqrt{1-e^2}\sin i} F_z$$

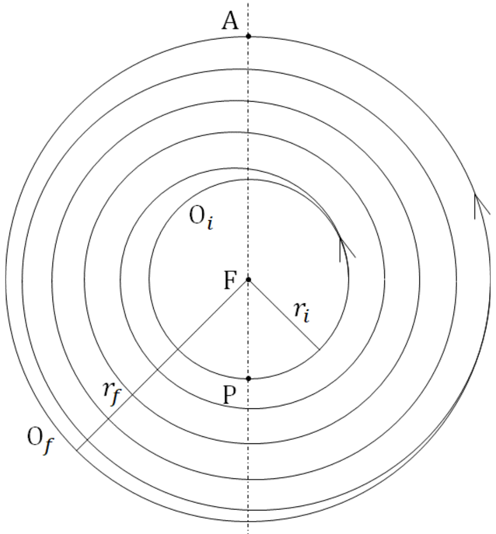
$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\sqrt{1-e^2}}{nae}\left[-\cos f\, F_r + \sin f\left(1+\frac{r}{p}\right) F_\theta\right] - \frac{r\cot i \sin(\omega+f)}{h} F_z$$

$$\frac{\mathrm{d}M_0}{\mathrm{d}t} = \frac{1}{na^2 e}\left[(p\cos f - 2er)F_r - (p+r)\sin f\, F_\theta\right] - \frac{\mathrm{d}n}{\mathrm{d}t}(t-t_0)$$

One example is the application of continuous micro-thrust to orbit transitions.
For **the Hohmann transfer** (upper right figure), $\Delta V = \Delta v_1 + \Delta v_2$.
In **the spiral transition** (lower right figure), $\Delta V = v_i - v_f$.

In an orbit transfer from a 200 km altitude circular orbit to GEO, the spiral transition ($\Delta V = 4.71\mathrm{km/s}$) has a higher dV than the Hohmann transfer ($\Delta V = 3.93\mathrm{km/s}$), but **less propellant is required if electric propulsion with a high specific impulse is used**.





2.2 ΔVEGA

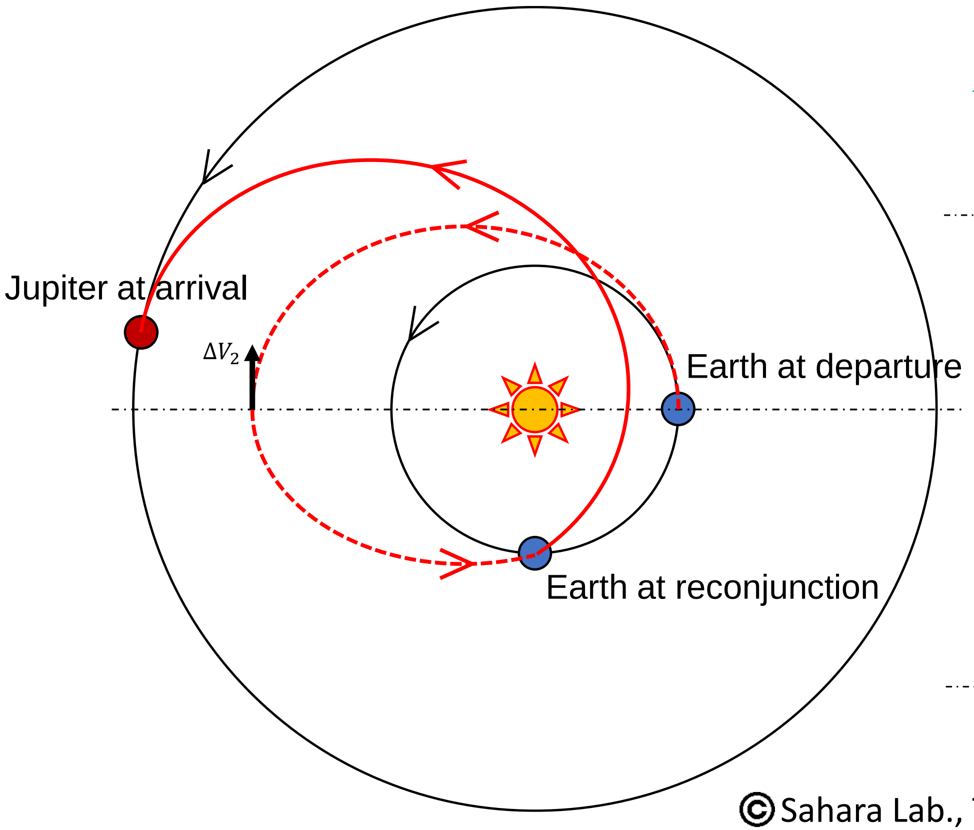
If a spacecraft departing from Earth experiences a velocity change outside of Earth's sphere of influence, it can perform **a swing-by using Earth at the time of reconjunction**.

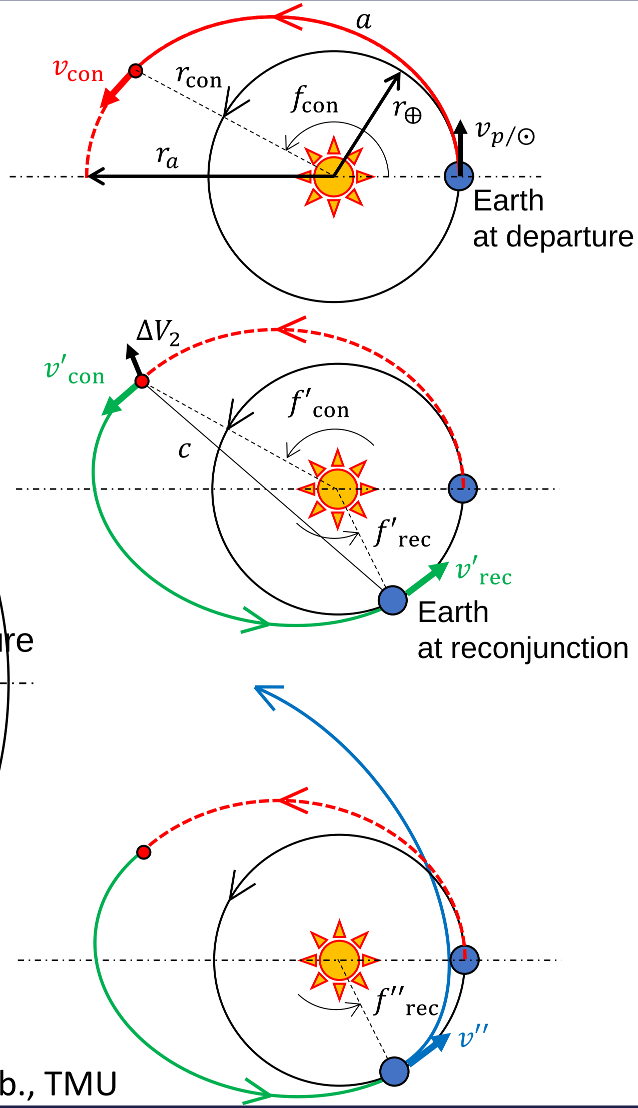
When electric propulsion is used, it is called **EPΔVEGA**.

For example, by adjusting a, f_{con}, $r_{p/\oplus}$, T_f, and $r'_{p/\oplus}$ well, the orbit is designed to reach the target planet.

Hohmann transfer achieves the transfer with minimum energy when $r_f/r_i < 11.9$, but using **ΔVEGA**, the required ΔV can be smaller than that of the Hohmann transfer because the Earth's revolution energy can be used, although it takes time in the order of years to reach the target.

For example, to reach Jupiter from an Earth orbit of 200 km altitude, the Hohmann transition requires $\Delta V = 6.30$ km/s, while $\Delta V = 5.30$ km/s with ΔVEGA.





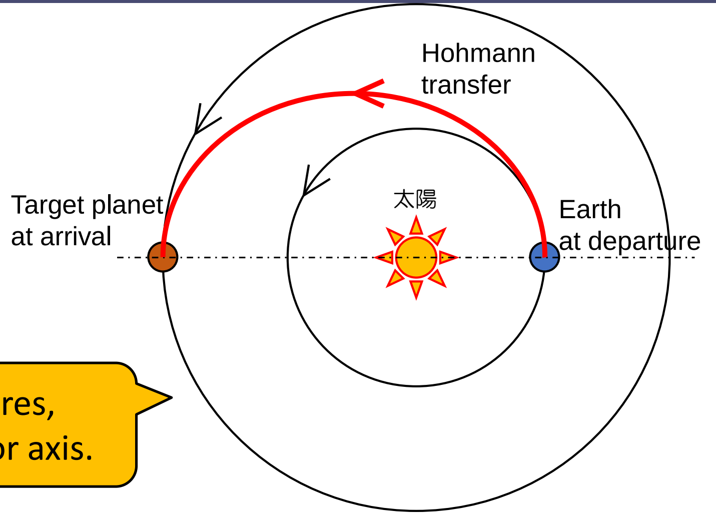
Lambert’s Problem

Finds the unique orbit connecting two points, when given a departure point $\mathbf{r}_D$, an arrival point $\mathbf{r}_A$, and a flight time T_f between them.

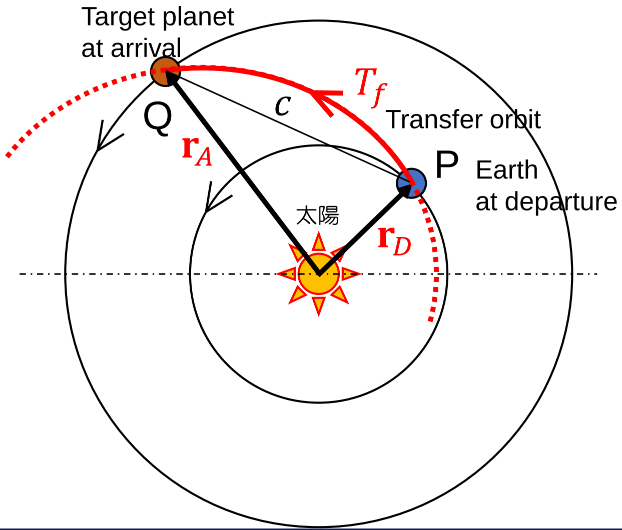
Hohmann transfer gives the smallest ΔV orbital transfer between circular orbits where the Earth at departure, the Sun, and the target planet at arrival are aligned.

What if this alignment does not hold?

As is clear from the upper and lower right figures, the Hohmann orbit has the smallest semimajor axis.



There are an infinite number of orbits connecting two points.
Lambert's theorem states that given a time of flight, the transfer orbit is **uniquely determined**.



Lambert's Theorem

The time of flight of an orbit connecting two points is uniquely determined by its semimajor axis, a, $|\mathbf{r}_D| + |\mathbf{r}_A|$, and the linear distance, c, between the two points.
Note that it depends on the sum of the absolute values of $\mathbf{r}_D$ and $\mathbf{r}_A$, not on each of them.

If you can solve the Lambert's problem in addition to the orbit mechanics discussed in the previous chapter, you will be able to design any orbit.

Hill's Equation

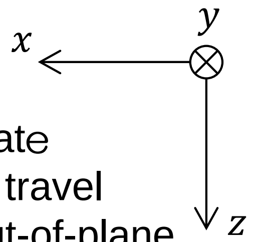
Describes the motion of a chaser (e.g., spacecraft) in the vicinity of a target (e.g., space station) in orbital motion with angular velocity $\boldsymbol{\omega}$.

Adopted LVLH coordinate system, it is described as follows:

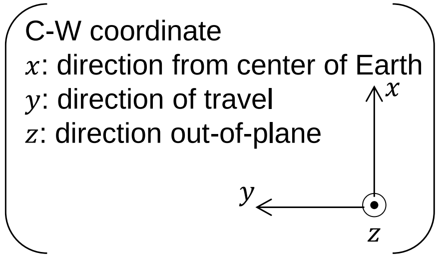
$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 2n\dot{z} + F_x \\ -n^2 y + F_y \\ 3n^2 z - 2n\dot{x} + F_z \end{bmatrix}$$

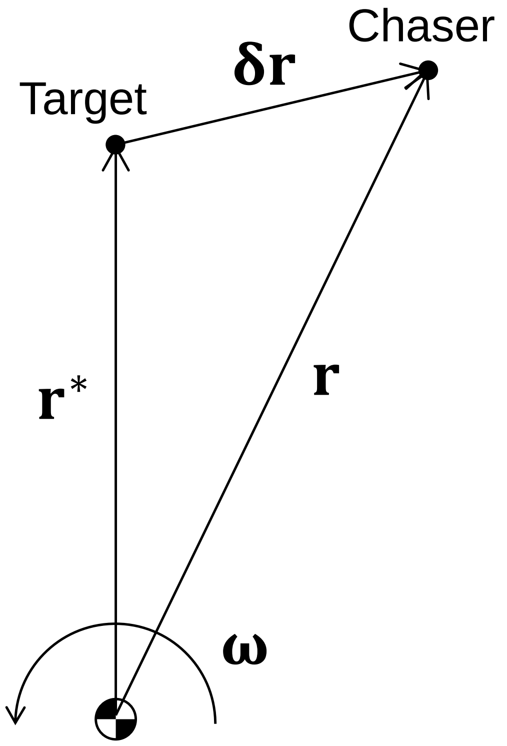
y is independent and single-oscillating.

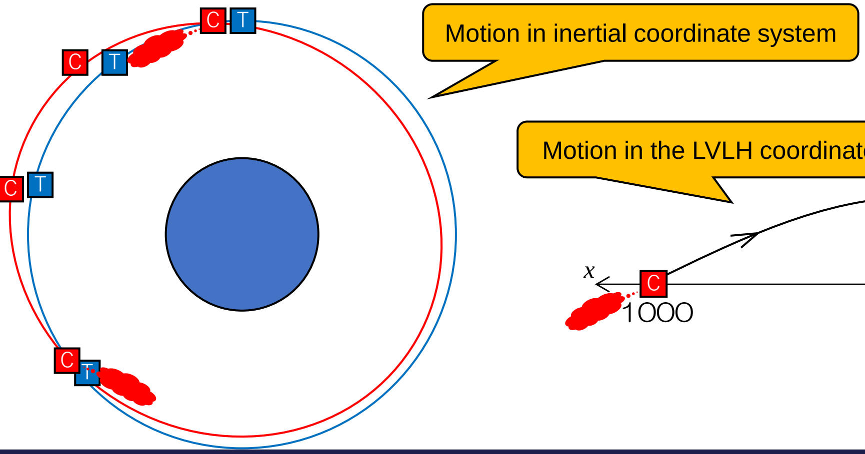
x and z are coupled. For example, to catch up with a forward target, both the direction of altitude and the direction of travel must be controlled.

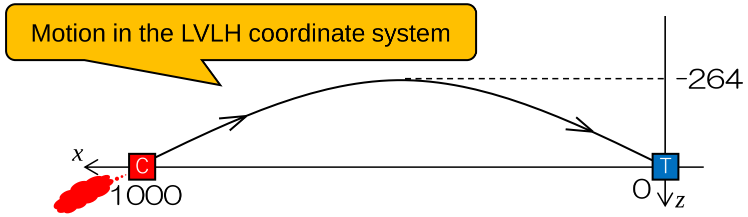


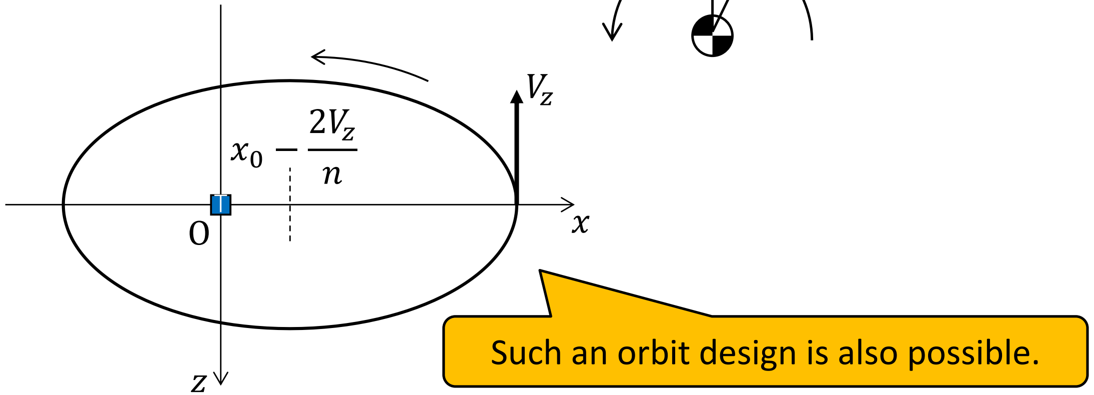
LVLH coordinate
x: direction of travel
y: direction out-of-plane
z: direction to center of Earth







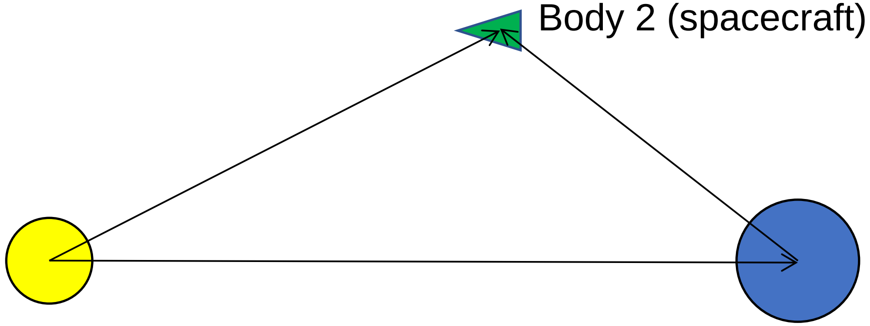




Poincaré's theorem

"When perturbations are added to an integrable system, the system generally becomes non-integrable.“
Therefore, the three-body problem is generally not solvable analytically, and its solution must be done numerically.

When $m_2 \ll m_1, m_3$ for one of the three bodies (spacecraft, m_2), the trajectory of the spacecraft moving in the gravity field of the other two bodies (Body 1 and Body 3) can be obtained analytically.
This is called **the restricted three-body problem**.



When two bodies, excluding the spacecraft, are in circular motion, they can be treated as stationary in a co-rotating system. This case is called **the circular restricted three-body problem**.

Since this is a co-rotating system, **inertia terms** as centrifugal force and Coriolis force appear in the equations of motion.

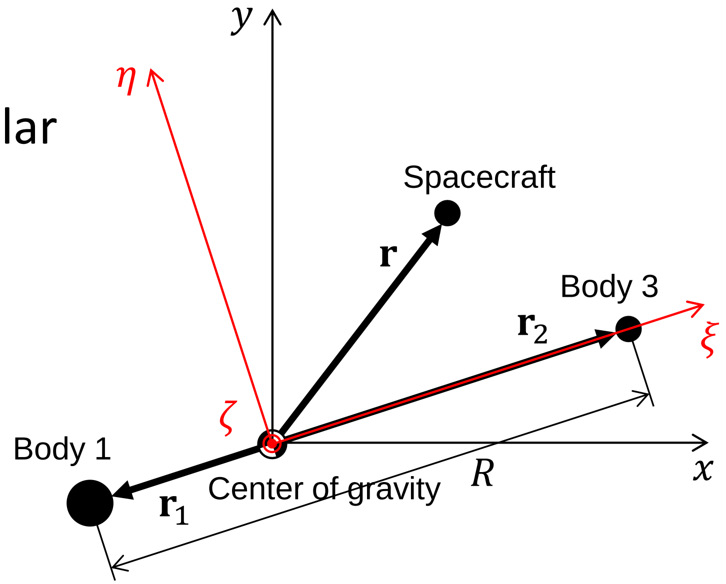
Lagrange Point

Equations of motion of a spacecraft in a rotating system as the dimensionless circular restricted three-body problem is as follows:

$$\ddot{\xi} - 2\dot{\eta} = \frac{\partial W}{\partial \xi}$$

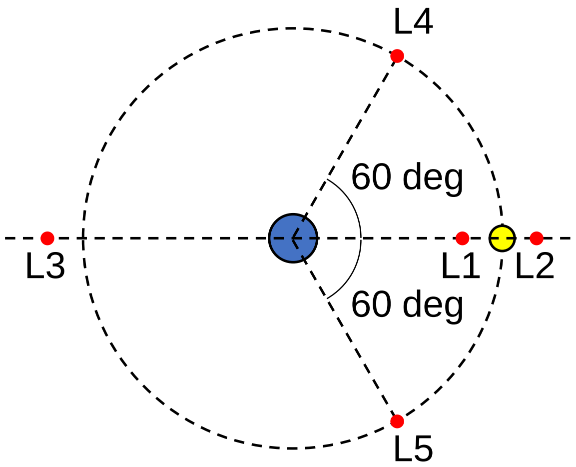
$$\ddot{\eta} + 2\dot{\xi} = \frac{\partial W}{\partial \eta}, \quad W = \frac{1}{2}(\xi^2 + \eta^2) + \frac{m_1}{|\mathbf{r}-\mathbf{r}_1|} + \frac{m_2}{|\mathbf{r}-\mathbf{r}_2|}$$

$$\ddot{\zeta} = \frac{\partial W}{\partial \zeta}$$



In a rotating system, there exist Lagrangian points L1 to L5 satisfying the following conditions where the position of the spacecraft does not change due to the balance between centrifugal force and gravity.

In the case of the Earth-Moon system, these are called
EML1: cis-lunar, EML2: trans-lunar, EML3: trans-Earth, EML4 & EML5: trojan point.



Lyapunov orbit

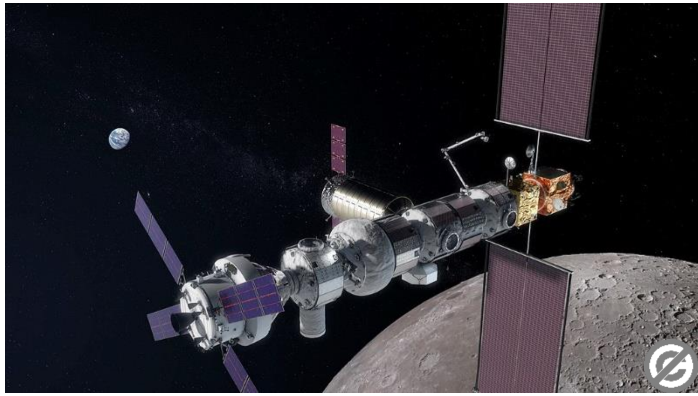
In the vicinity of L1 and L2, there exist periodic orbits that are Lyapunov stable in-plane or out-of-plane.
This is called a Lyapunov orbit.
https://www.youtube.com/watch?v=I3MNOTNMIa8

Distant Retrograde Orbit, DRO

Highly stable periodic orbits exist that retrograde in-plane around the secondary object.
https://youtu.be/X5O77OV9_ek?t=28

Halo Orbit / Lissajous Orbit

A family of Lyapunov orbits in the plane yields periodic orbits with periodicity in the out-of-plane direction at a certain bifurcation point. This is called a halo orbit. Because the halo orbit is unstable, a small ΔV is required to maintain the orbit.
A halo orbit with an extremely large uniaxial direction is called Near Rectilinear Halo Orbit (NRHO), and its adoption by the Lunar Orbital Platform-Gateway (LOP-G) is being considered.
https://www.youtube.com/watch?v=X5O77OV9_ek
If the orbit is aperiodic, it is called Lissajous orbit.



Lunar Orbital Platform-Gateway, LOP-G
https://en.wikipedia.org/wiki/Lunar_Gateway

3. Application to Satellite Design

This chapter introduces applications of orbital mechanics to satellite design.

Tsiolkovsky rocket equation

Let m_i and m_f be the masses before and after injection, respectively.
Acceleration of gravity is g and the specific impulse is I_{SP}, then the exhaust velocity is expressed as gI_{SP}.
The velocity increment obtained by this injection is

$$\Delta V = gI_{SP} \ln \frac{m_i}{m_f},$$

which is called the Tsiolkovsky rocket equation.

Example

Find the incremental velocity that can be obtained by a spacecraft with the initial mass of 4,000 kg, a hydrazine monopropellant propulsion system (specific impulse of 200 s), and the propellant mass of 1,000 kg.

Answer

As the final mass is 4,000 – 1,000 = 3,000 kg, $\Delta V = 9.8 \times 200 \times \ln 4000/3000 = 564\ \mathrm{m/s}$.

The objectives of orbit design are (1) To find an orbit that will allow the mission to be completed, and (2) To obtain the necessary ΔV and deliver it to the system engineer.

Therefore, **orbit mechanics and system design are connected by the Tsiolkovsky rocket equation!!**

(1) Low Earth Orbit (LEO) to GEO

Using an MMH/NTO propulsion system with a specific impulse of 300 s, find the propellant mass required for the transfer from a circular orbit of 200 km altitude to GEO with an initial mass of 4,000 kg. However, do not consider orbital plane change.

Answer

The velocity increment required for this orbit transfer is 3.93 km/s based on the Hohmann transfer.
Therefore,

$$m_f = 4000 \times \exp\left(-\frac{3930}{9.8 \times 300}\right) = 1051 \quad \therefore m_p = m_i - m_f = 2949 \text{ kg}$$

Here, if we can use a Hall thruster with a specific impulse of 2,000 s,

$$m_f = 4000 \times \exp\left(-\frac{4710}{9.8 \times 2000}\right) = 3146 \quad \therefore m_p = m_i - m_f = 854\text{kg}$$

Orbit mechanics indicates that a higher specific impulse requires much less propellant.

However, the following considerations must be taken into account **when designing a satellite as system**: (1) The required time is on the order of years due to the spiral transfer; (2) The time to pass through the Van Allen belt is extremely long; (3) Although only a small amount of propellant is required, what if the weight of the power source is also taken into account (thrust to power ratio)?; (4) Is it probable to handle high-pressure gases such as Xenon or Krypton?

3. Application to Satellite Design

3.2 Examples – (1) LEO to GEO, (2) Phase shift

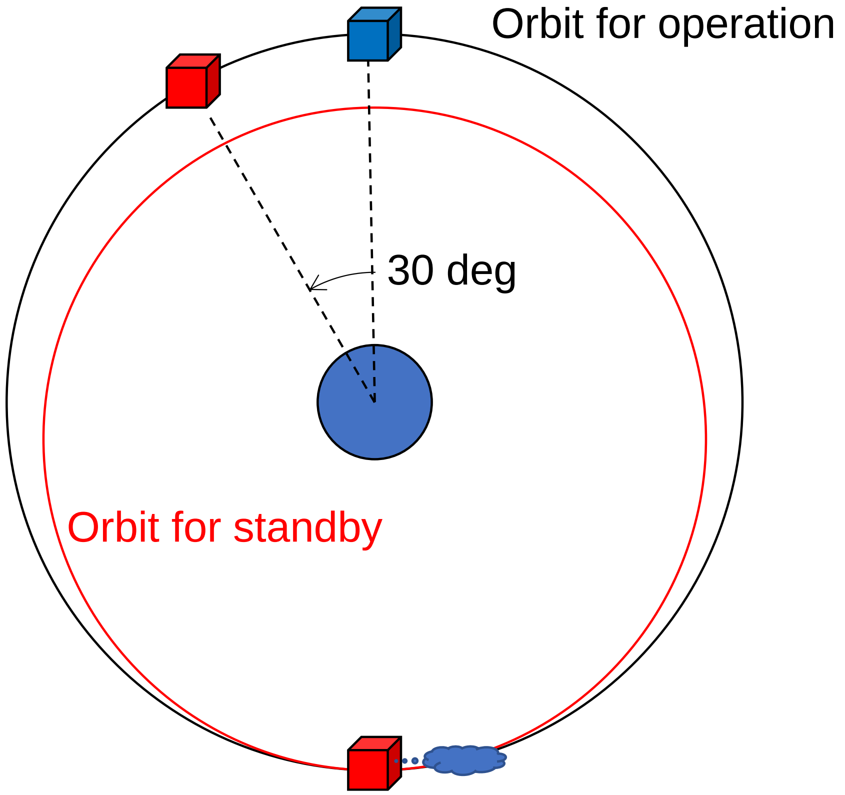
(2) Phase shift

Two satellites were injected into a circular orbit at an altitude of 800 km by Rideshare.
However, due to communication bandwidth limitations, the two satellites must be in phase by 30 degrees in orbit.
Therefore, one of them performs a 30-degree phase shift.

An example of solution

One satellite is decelerated at a point in the orbit for operation and shifted to an orbit for standby with a reduced orbit length radius.

When the phase difference between the two satellites increases due to the difference in orbital period, the satellite is accelerated at a injection point and returns to its original orbit.



4. Conclusion

4. Conclusion

By mastering **Chapter 1**, you have acquired a good foundation in orbital mechanics.

If you understand the Lambert and rendezvous problems in **Chapter 2**, you can be proud to say that you are an intermediate student of orbital mechanics.

If you are able to use everything in Chapter 2, you will already be in an important position to teach orbital mechanics.

The mathematics used in orbital mechanics is not very difficult, and the physical phenomena are very straightforward.

On the other hand, the orbits that you design will become more and more beautiful, depending on your ideas.

We hope you will enjoy orbital mechanics and find your own wonderful orbits♥♥♥

Thank you very much.

[Disclaimer]
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