Analysis of small-scale magnetic field generation in mhd-shell model

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Two approaches in mhd-dynamo

Dynamo is a branch of physics dedicated to the study of self-generation of magnetic fields in random media. We consider one of the modes of operation of the dynamo, which is called the small-scale or turbulent dynamo. A small-scale dynamo is described in a standard way by the Kazantsev model, which is obtained from the magnetic induction equation by averaging over the velocity field [Kazantsev, 1967].

$$\partial_t \mathbf{B} = \mathrm{rot}\, \mathbf{v} \times \mathbf{B} + \eta \Delta \mathbf{B}$$

In this work, we compared the processes of dynamo generation in two models: in the shell model and the Kazantsev model. Comparisons were made at the initial times, where the effects of the influence of a random magnetic field on the velocity field are insignificant. The purpose of this comparison is related to the fact that both models have their advantages and disadvantages.

Shell model

Advantages	Disadvantages
Energy spectrum corresponding to realistic turbulence	The discrete spectrum of wave numbers is considered
Nonlinear terms are taken into account	Abstract collective variables are introduced.

Kazantsev model

Advantages	Disadvantages
Can be derived directly for short-correlated flow	This model is linear
Generalized to anisotropic turbulence.	Correlation times are the same on all scales

Kazantsev model

Kazantsev's model, which describes the growth of magnetic energy in a random velocity field, can be obtained by averaging the magnetic induction equation over a random velocity field.

$$\partial_t \mathbf{B} = \mathrm{rot}\, \mathbf{v} \times \mathbf{B} + \eta \Delta \mathbf{B}$$

The correlation tensors in mirror-symmetric isotropic turbulence have the form

$$\langle B_i(x_1, t_1) B_j(x_2, t_2) \rangle = \left(M + \frac{r}{2} \frac{\partial M}{\partial r} \right) \delta_{ij} - \frac{r_i r_j}{2r} \frac{\partial M}{\partial r}$$

$$\langle v_i(x_1, t_1) v_j(x_2, t_2) \rangle = \left(F + \frac{r}{2} \frac{\partial F}{\partial r} \right) \delta_{ij} - \frac{r_i r_j}{2r} \frac{\partial F}{\partial r}$$

The averaging of the magnetic induction equation was first done by Kazantsev for a flow delta-correlated in time. The input parameter to this equation is turbulent diffusion, where Rm is the magnetic Reynolds number and τ is the correlation time.

$$\boxed{\frac{\partial M}{\partial t} = \frac{2}{r^4} \frac{\partial}{\partial r} \left(r^4 \eta \frac{\partial M}{\partial r} \right) + \frac{2M}{r^4} \frac{\partial}{\partial r} \left(r^4 \frac{\partial \eta}{\partial r} \right)}$$

$$\eta(r) = \frac{1}{Rm} + \tau \frac{F(0) - F(r)}{3}$$

Shell model

Shell models describe the transport of energy and helicity on a finite number of spectral shells. The formalism of shell models was proposed by Obukhov for hydrodynamic type systems.

1. in the non-dissipative limit, the system conserves the phase volume;
2. the system has at least one quadratic integral of motion;
3. equations contain quadratic non-linearity;
4. when considering long chains of equations, the latter are limited to local interactions, that is, only nearest neighbors interact.

Conservation laws

$$E = E_V + E_B = \int (\mathbf{v}^2 + \mathbf{B}^2) dV$$

$$H_B = \int (\mathbf{A} \cdot \mathbf{B}) dV$$

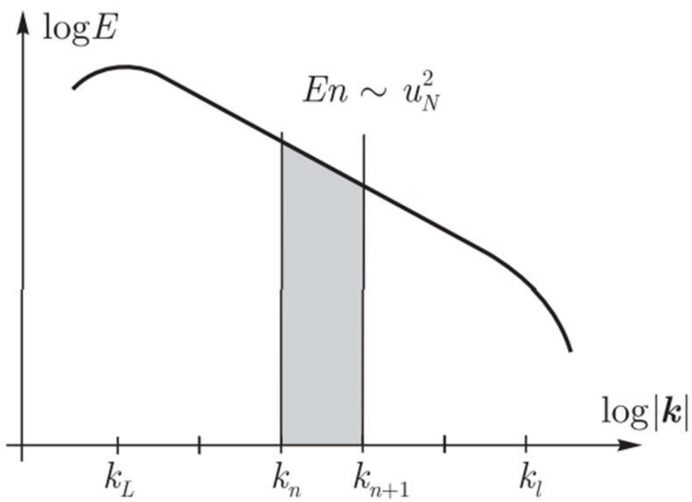
$$H = \int (\mathbf{v} \cdot \mathbf{B}) dV$$

One of the types of hydrodynamic type is the system of equations of magnetohydrodynamics

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = (\mathbf{B} \cdot \nabla)\mathbf{B} - \nabla(P + B^2/2) + Re^{-1}\Delta\mathbf{v}$$

$$\partial_t \mathbf{B} + (\mathbf{v} \cdot \nabla)\mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{v} + Rm^{-1}\Delta\mathbf{B}$$

Shell model



In the work 41 spectral shells were considered, where $k_n<|\mathbf{k}|<k_{n+1}$, where $k_n=1/\lambda^n$. Each of which is characterized by collective complex variables of the velocity field U_n and magnetic field B_n. Shell model that satisfies the conservation laws corresponding to the three-dimensional case of ideal magnetohydrodynamics has the following form

$$d_tU_n = ik_n(\Lambda_n(U,U) - \Lambda_n(B,B)) - \frac{k_n^2 U_n}{Re}$$

$$d_tB_n = ik_n(\Lambda_n(U,B) - \Lambda_n(B,U)) - \frac{k_n^2 B_n}{Rm}$$

$$\Lambda_n(X,Y) = q^2(X_{n+1}Y_{n+1} + X^*_{n+1}Y^*_{n+1}) - X^r_{n-1}Y_n - \\ -X_nY^r_{n-1} + iq(2X^*_nY^i_{n-1} + X^r_{n+1}Y^i_{n+1} - X^i_{n+1}Y^r_{n+1}) + \\ +X_{n-1}Y_{n-1} + X^*_{n-1}Y^*_{n-1} - q^2(X^r_{n+1}Y_n + X_nY^r_{n+1}) + \\ +iq(2X^*_nY^i_{n+1} + X^r_{n-1}Y^i_{n-1} - X^i_{n-1}Y^r_{n-1})$$

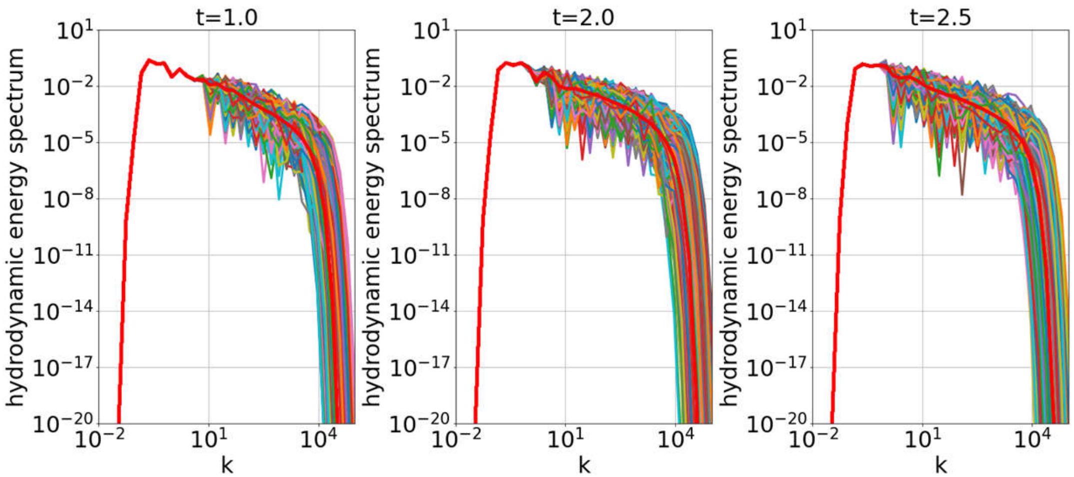
Conservation laws

$$E^T = \sum(|U_n|^2 + |B_n|^2)/2$$

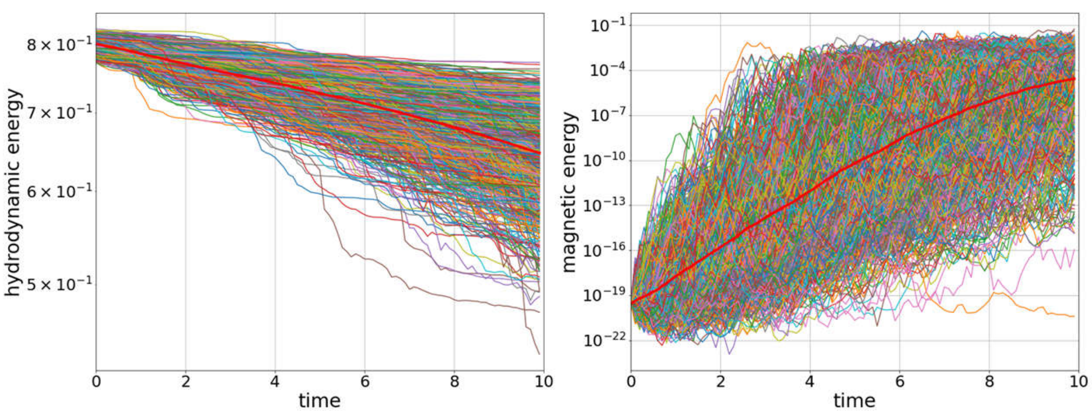
$$H_m = \sum ik_n^{-1}((B^*_n)^2 - B_n^2)/4$$

$$H = \sum(U_nB^*_n + B_nU^*_n)/2$$

Shell model. Numerical results.



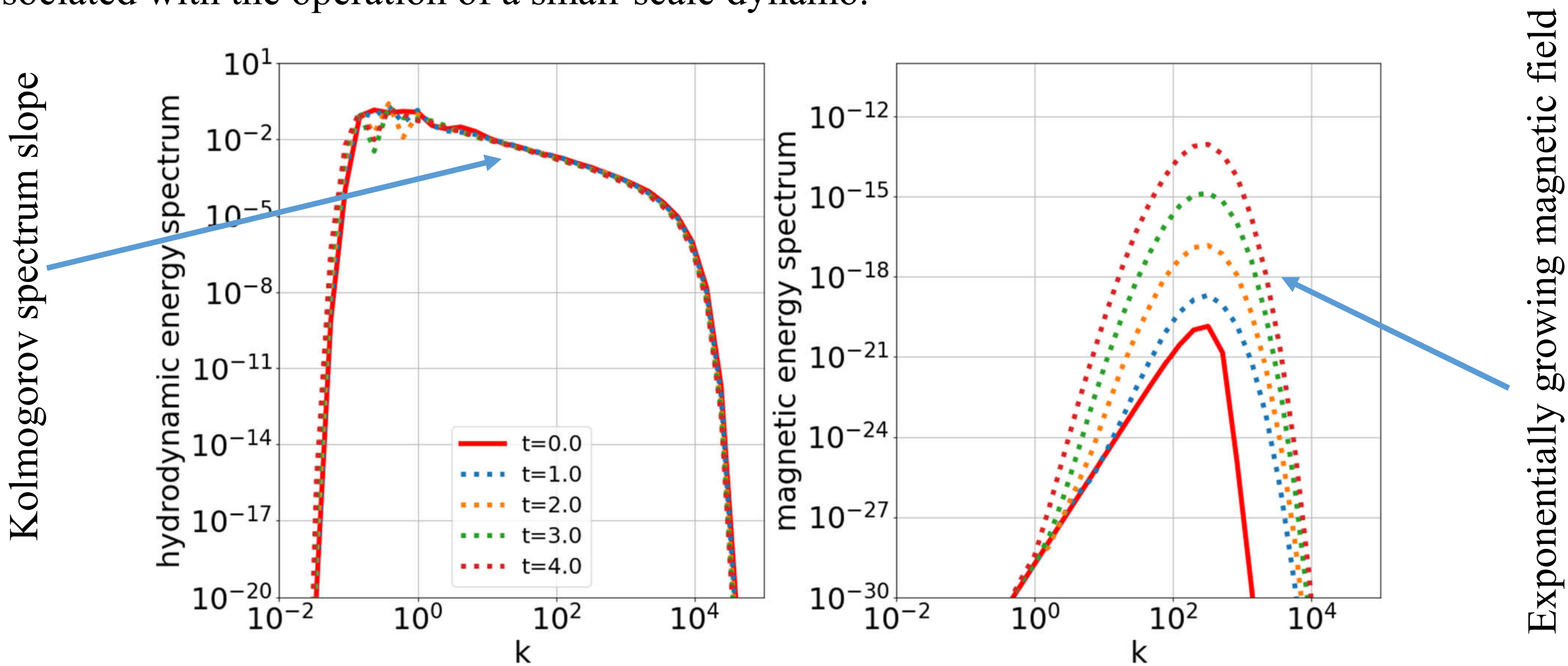
In the absence of a magnetic field, the hydrodynamic energy spectrum takes the form of a -5/3 Kolmogorov spectrum. The graphs show 1024 implementations at different times with 1% initial noise. The bold red line shows the average spectrum.



When the magnetic field is turned on, an energy exchange occurs, the hydrodynamic energy decreases, and the magnetic energy grows. The figure shows 1024 implementations of changes in hydrodynamic and magnetic energy over time. The red line shows their average values. At small times from 0 to 4, we observe an exponential growth of the magnetic energy, which we will investigate.

Shell model. Numerical results.

The hydrodynamic spectrum actually does not change over the considered time interval. The spectrum of the magnetic field shows an increase in magnetic energy on a characteristic scale of $10^{2.5}$ with a characteristic rate of 4.3. The small scale and high growth rates suggest that this growth is associated with the operation of a small-scale dynamo.



Comparison of two models

To check whether this growth is the work of a small-scale dynamo, a similar numerical experiment was carried out for the Kazantsev model. The correlation function $F(r)$ was reconstructed from the hydrodynamic spectrum, and turbulent diffusion was calculated from this function. Then, the evolution of the magnetic correlation function was reconstructed in the Kazantsev model. According to the function $M(r)$, the magnetic energy density was calculated, which was compared with the energy density obtained in the shell model. **Rm = 10000.**

$$F(r) = \frac{1}{4\pi^3}\int_0^\infty E_V(k)\frac{\sin(kr) - kr\cos(kr)}{(kr)^3}dk \rightarrow \eta(r) \rightarrow M(r,t) \rightarrow E_B(k) = 4\pi\int_0^\infty M(r)kr\left(\sin(kr) - kr\cos(kr)\right)dr$$

$$F(r) = \frac{1}{4\pi^3}\int_0^\infty E(k)\sum_{i=0}^\infty (-1)^{i+1}\frac{2i(kr)^{2i-2}}{(2i+1)!}dk \Rightarrow \eta(r) = \frac{1}{Rm} + \tau\frac{F(0) - F(r)}{3}$$

$$F_r(r) = \frac{1}{4\pi^3}\int_0^\infty E(k)\sum_{i=0}^\infty (-1)^{i+1}\frac{(4i^2 - 4i)k^{2i-2}r^{2i-3}}{(2i+1)!}dk \Rightarrow \eta_r(r) = \tau\frac{-F_r(r)}{3}$$

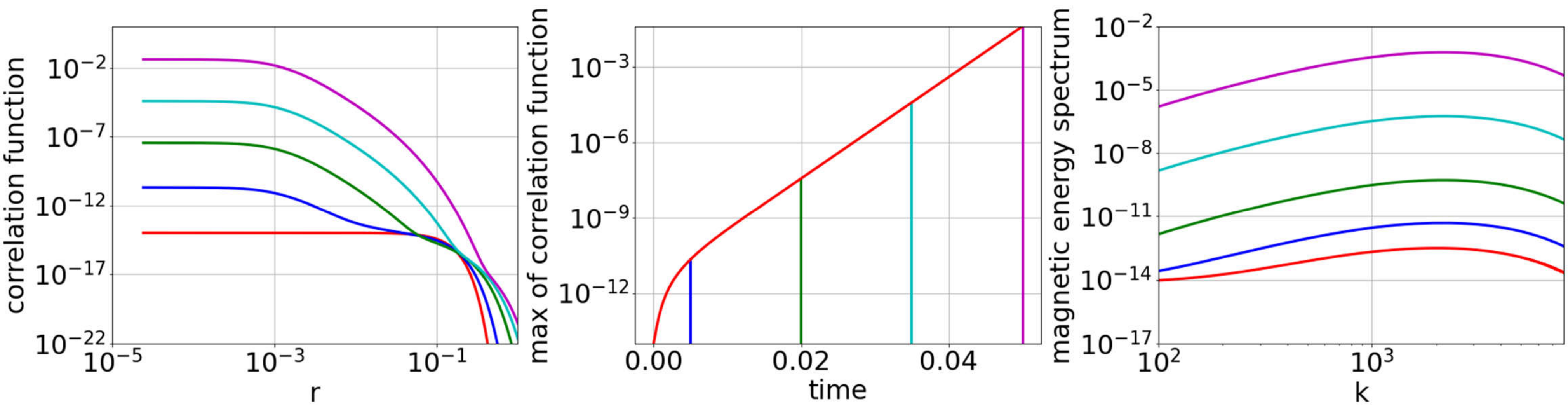
$$F_{rr}(r) = \frac{1}{4\pi^3}\int_0^\infty E(k)\sum_{i=0}^\infty (-1)^{i+1}\frac{(4i^2 - 4i)(2i-3)k^{2i-2}r^{2i-4}}{(2i+1)!}dk \Rightarrow \eta_{rr}(r) = \tau\frac{-F_{rr}(r)}{3}$$

Comparison of two models

These graphs were obtained for the hydrodynamic energy spectrum of the interval $0 < k < 2$, $\tau = 10^{3,5}$, $\mathrm{Rm} = 10000$, $\mathrm{T} = 1000$, $\mathrm{dT} = 0.002$

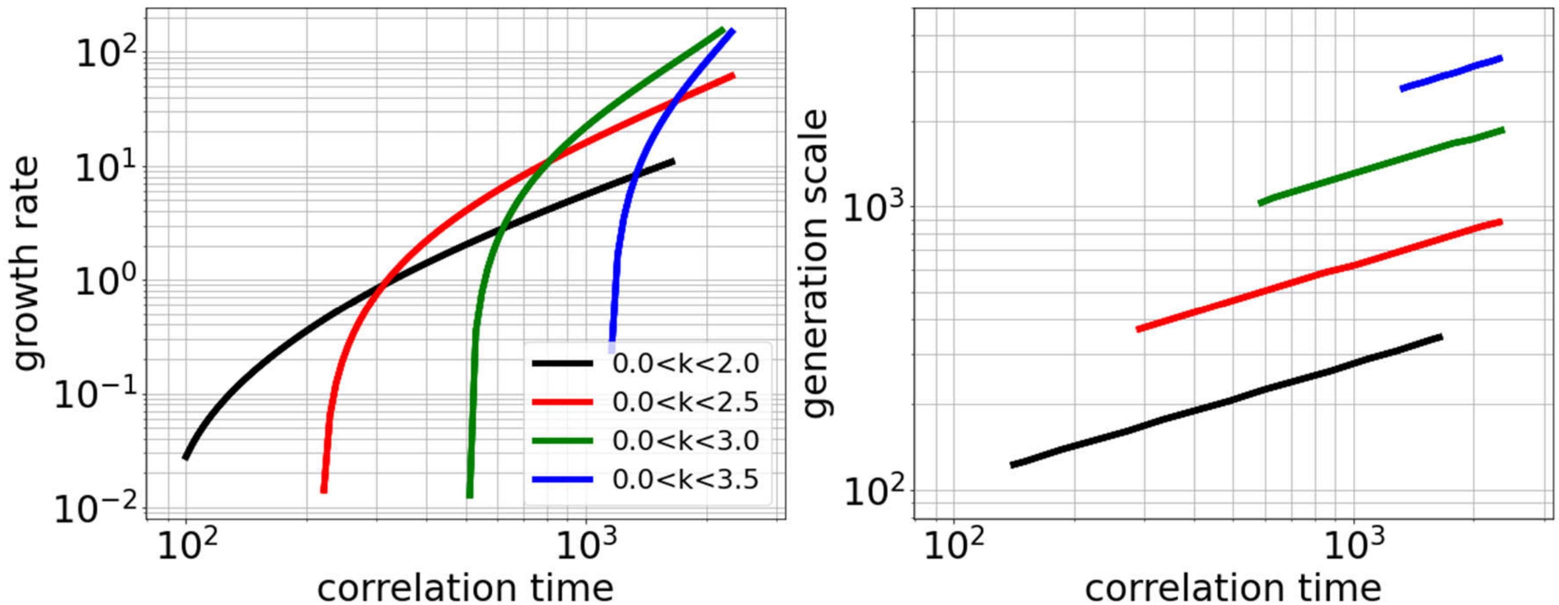
$N_r = 500$ $\quad\quad$ $N_r = 100000, N_k = 100$

$$\frac{\partial M}{\partial t} = \frac{2}{r^4}\frac{\partial}{\partial r}\left(r^4 \eta \frac{\partial M}{\partial r}\right) + \frac{2M}{r^4}\frac{\partial}{\partial r}\left(r^4 \frac{\partial \eta}{\partial r}\right) \quad \Rightarrow \quad E_B(k) = 4\pi \int_0^\infty M(r) kr \left(\sin(kr) - kr\cos(kr)\right) dr$$



Correlation time

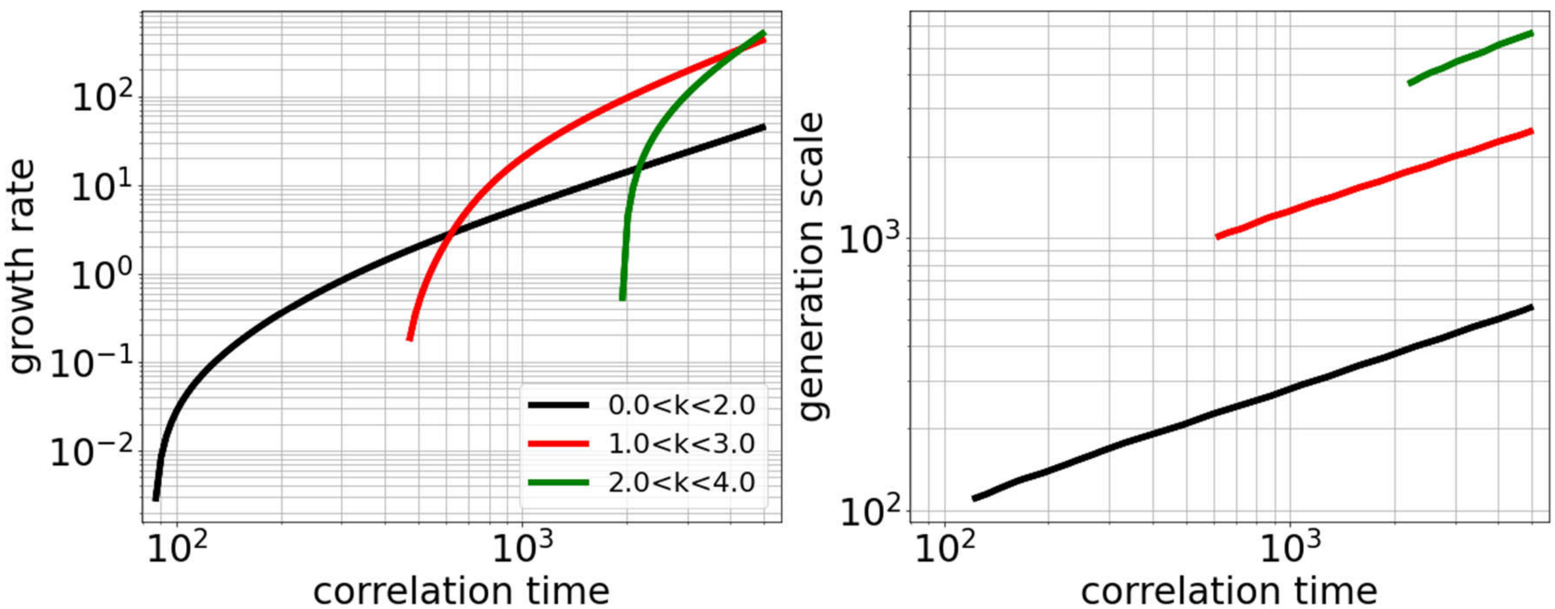
The correlation time is not included in the shell model, but is included in the Kazantsev model, so we varied the correlation time and compared the scales on which the generation and the rates of this generation occurs. If we consider the entire spectrum, then the generation threshold turns out to be high and the generation rates turn out to be extremely high.



If we assume that a large-scale dynamo is responsible for the operation of a small-scale dynamo, then the generation threshold, generation rate, and generation scale decrease.

Correlation time

A similar situation is observed if we do not reduce the considered range, but shift the considered range. For smaller-scale processes, the generation rate, the generation threshold, and the generation scale increase.



If we assume that both models describe the same process, then we can say that the large-scale part of the hydrodynamic spectrum is responsible for the small-scale generation.

Conclusions

- A comparison of the Kazantsev and Shell models showed that if we take the entire hydrodynamic spectrum, then it is impossible to choose such a correlation time that the generation rates and generation scales in the two models coincide.

- If we take the large-scale part of the hydrodynamic spectrum, it is possible to choose such a correlation time that the generation scales and generation rates in the two models will coincide.

- If we take the developed hydrodynamic spectrum up to the dissipative scale, then the generation rates in the Kazantsev model turn out to be much higher than the generation rates in the shell model, the generation scales turn out to be much smaller than in the shell model, and the generation threshold turns out to be extremely high.

- Thus, the large-scale part of the hydrodynamic spectrum is responsible for the operation of the small-scale dynamo.